

On the Theory of Ends of a Pro- p Group

By Kay Wingberg

Abstract. We study the group $H^1(G, \mathbb{F}_p[[G]])$ of ends of a pro- p group G and prove a pro- p analog of Stallings' decomposition theorem.

One of the most important results in the theory of abstract groups is Stallings' decomposition theorem [16]. Let $e(G) = 1 + \dim_{\mathbb{F}_2} H^1(G, \mathbb{F}_2[G])$ be the number of ends of an infinite finitely generated abstract group G . Then $e(G) = 1, 2$ or ∞ , and $e(G) = 2$ if and only if G has a subgroup of finite index which is isomorphic to \mathbb{Z} (Hopf 1943). If G is torsion-free, then the theorem of Stallings (1968) asserts that $e(G) = \infty$ if and only if G is a free product of non-trivial subgroups. Up to now it was impossible to obtain an analog of this theorem for profinite groups. But we will see that for pro- p groups the situation is much better.

If p is a fixed prime number and G a pro- p group, then Λ_G denotes the completed group ring $\mathbb{F}_p[[G]]$ of G over \mathbb{F}_p and I_G is its augmentation ideal, i.e. the kernel of the augmentation map $\Lambda_G \twoheadrightarrow \mathbb{F}_p$. Consider the (continuous) cohomology groups $H^i(G, \mathbb{F}_p[[G]])$ and let

$$h^i(G) = \dim_{\mathbb{F}_p} H^i(G, \mathbb{F}_p[[G]]).$$

The number

$$e(G) = 1 - h^0(G) + h^1(G)$$

is called the *number of ends* of G . If G is infinite, then $e(G) = 1 + h^1(G)$.

We denote the coinvariants of a left (resp. right) Λ_G -module M by $M_G = M/I_G M$ (resp. $M_G = M/M I_G$). Considering the right Λ_G -module $H^1(G, \mathbb{F}_p[[G]])$, let

$$f(G) = \dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p[[G]])_G.$$

For pro- p groups first results were obtained by Korenev [9]. He proved the following analog of Hopf's theorem:

Theorem 1: *Let G be a pro- p group, then*

- (i) $e(G) = 0, 1, 2$ or ∞ .
- (ii) $e(G) = 0$ if and only if G is finite.
- (iii) $e(G) = 2$ if and only if G has an open subgroup isomorphic to \mathbb{Z}_p .

In particular, if G is infinite, then $h^1(G) = 0, 1$ or ∞ , and if G is torsion-free, then $h^1(G) = 1$ if and only if $G \cong \mathbb{Z}_p$.

We call a non-trivial pro- p group G **freely decomposable** if it is the free pro- p product of s non-trivial closed subgroups with $s > 1$; otherwise G is called **freely indecomposable**. By $s(G)$ we denote the number of freely indecomposable factors of G (which is well-defined, see theorem (2.1) below).

We will give an alternative short proof of Korenev's theorem, but our main result will be the following pro- p analog of Stallings' theorem.

Theorem 2: *Let G be a finitely generated pro- p group. Then the following two assertions are equivalent:*

- (i) G is freely decomposable.
- (ii) I_G is decomposable as left $\mathbb{F}_p[[G]]$ -module.

Furthermore, the following are equivalent:

- (iii) *There exists an open subgroup H of G such that every open subgroup of H is freely decomposable.*
- (iv) $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p[[G]]) = \infty$.

If, in addition, G is torsion-free, then all four statements are equivalent.

Corollary 1: *Let G be a finitely generated torsion-free pro- p group. Then the following assertions hold:*

1. *If G is not free, then G is the free pro- p product of $s(G) = f(G) + 1$ freely indecomposable closed subgroups; furthermore, the $\mathbb{F}_p[[G]]$ -module $H^1(G, \mathbb{F}_p[[G]])$ is free of rank $f(G)$. If G is free, then $s(G) = f(G)$ and there is an exact sequence $0 \rightarrow \mathbb{F}_p[[G]] \rightarrow \mathbb{F}_p[[G]]^{f(G)} \rightarrow H^1(G, \mathbb{F}_p[[G]]) \rightarrow 0$.*
2. *Let H be an open subgroup of G . Then*

$$s(H) = (G : H)(s(G) - 1) + 1.$$

In particular, G is freely indecomposable if and only if H is, and I_G is indecomposable as left $\mathbb{F}_p[[G]]$ -module if and only if I_H is indecomposable.

We recall the notion of a pro- p duality group [12] (3.4.6): For a discrete G -module A and $i \geq 0$, let

$$D_i(G, A) = \varinjlim_U H^i(U, A)^\vee,$$

where $^\vee$ denotes the Pontryagin-dual, the direct limit is taken over all open subgroups U of G and the transition maps are the duals of the corestriction maps. $D_i(G, A)$ is a discrete G -module in a natural way. Assume that the cohomological dimension $\mathrm{cd}_p G = n$ is finite. Then the G -module

$$D(G) = \varinjlim_{\nu \in \mathbb{N}} D_n(G, \mathbb{Z}/p^\nu \mathbb{Z})$$

is called the dualizing module of G and we have in a natural way the trace map

$$\mathrm{tr} : H^n(G, D(G)) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p.$$

The pro- p group G is called a **duality group of dimension n** if for all $i \in \mathbb{Z}$ and all finite p -primary G -modules A , the cup-product and the trace map

$$H^i(G, \mathrm{Hom}(A, D(G))) \times H^{n-i}(G, A) \xrightarrow{\cup} H^n(G, D(G)) \xrightarrow{\mathrm{tr}} \mathbb{Q}_p/\mathbb{Z}_p$$

yield an isomorphism $H^i(G, \mathrm{Hom}(A, D(G))) \cong H^{n-i}(G, A)^\vee$.

Corollary 2: *Let G be a finitely generated pro- p group of cohomological dimension $\mathrm{cd}_p G \leq 2$. Then G is the free pro- p product of finitely many duality groups and the following assertions are equivalent:*

- (a) G is a duality group of dimension 2.
- (b) G is freely indecomposable.
- (c) I_G is indecomposable as left $\mathbb{F}_p[[G]]$ -module.

In particular, if G is a 2-generator group with $\mathrm{cd}_p G = 2$, then G is a duality group.

Remarks:

1. If G is a finitely generated FAB pro- p group, i.e. the abelianization U^{ab} of every open subgroup U of G is finite, then, by the principal ideal theorem, $H^1(G, \mathbb{F}_p[[G]]) = 0$, see (1.3)(iii) below. One easily sees directly that G is freely indecomposable: suppose that $G = H_1 * H_2$ is a non-trivial decomposition and let U be the kernel of the map $G \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$, where each H_i surjects onto $\mathbb{Z}/p\mathbb{Z}$. By the pro- p analog of Kurosh' subgroup theorem U has a free factor of rank $r = p - 1$, hence U^{ab} surjects onto $(\mathbb{Z}_p)^r$, a contradiction.

2. In number theory an example of a class of pro- p groups satisfying the first assertion of corollary 2 was known before: let $G_S = \text{Gal}(k_S(p)|k)$ be the Galois group of the maximal p -extension of a number field k unramified outside the set S of primes of k . If $p \neq 2$, k contains the group of p -th roots of unity and S is finite and contains all primes above p and all archimedean primes, then $\text{cd}_p G \leq 2$ and G_S is a free pro- p product of duality groups, i.e. it is itself a duality group or a free pro- p product of decomposition groups and a free pro- p group, see [12] (10.9.8).

3. In an appendix we collect corresponding results in the case of abstract groups.

4. After finishing this paper we learned that Th. Weigel and P. A. Zalesskiĭ also proved an analogue of Stallings' decomposition theorem (for arbitrary finitely generated pro- p groups) using a slightly different definition for the group of ends, see [19]. Finally I would like to thank Thomas Weigel for pointing out an error in proposition (1.4) of an earlier version of this paper.

1 Decomposition of I_G

Let p be a fixed prime number, G a pro- p group and k a finite field of characteristic p . Then $\Lambda_G(k) = k[[G]]$ denotes the completed group ring of G over k with augmentation ideal $I_G(k)$; $\Lambda_G(k)$ is a local ring and indecomposable as $\Lambda_G(k)$ -module. Observe that a projective $\Lambda_G(k)$ -module is free. If $k = \mathbb{F}_p$, we write for short $\Lambda_G = \Lambda_G(\mathbb{F}_p)$ and $I_G = I_G(\mathbb{F}_p)$.

For the proof of the following result we refer to [18] prop.(2.1).

Proposition 1.1 *Let G be a pro- p group.*

- (i) *Let M be a non-zero indecomposable finitely generated left $\Lambda_G(k)$ -module. Then $\text{End}_{\Lambda_G(k)}(M)$ is a local ring.*
- (ii) *The Krull-Schmidt-Azumaya theorem holds, i.e. if M is a finitely generated left $\Lambda_G(k)$ -module, then M is expressible as a finite direct sum of indecomposable left submodules. Further, if*

$$M = \bigoplus_{i=1}^r M_i = \bigoplus_{j=1}^s N_j,$$

are two such sums, then $r = s$ and, by re-ordering, we have $M_i \cong N_i$, $i = 1, \dots, r$.

We introduce a notation which is used frequently in representation theory: write

$$N|M$$

to mean that the $\Lambda_G(k)$ -module N is isomorphic to a direct summand of the $\Lambda_G(k)$ -module M .

Corollary 1.2 *Let G be a pro- p group, N_1, N_2 and M left $\Lambda_G(k)$ -modules where M is non-zero, finitely generated and indecomposable.*

Then $M|(N_1 \oplus N_2)$ implies $M|N_1$ or $M|N_2$.

Proof: This is a formal consequence of the fact that $\text{End}_{\Lambda_G(k)}(M)$ is a local ring, see for example [10] (4.5). \square

Let $G^* = G^p[G, G]$ be the Frattini-subgroup of G , then

$$d(G) = \dim_{\mathbb{F}_p} G/G^* = \dim_{\mathbb{F}_p} H^1(G, \mathbb{Z}/p\mathbb{Z})$$

is the minimal number of generators of G . If $V \subseteq U$ are open normal subgroups of G , then

$$\text{Ver}_U^V: U/U^* \longrightarrow (V/V^*)^U$$

is the transfer map.

Let k be finite field and let M be a compact left $\Lambda_G(k)$ -module and N a compact $\Lambda_G(k)$ -bimodule. Then the group $\text{Hom}_{\Lambda_G(k)}(M, N)$ of continuous G -homomorphisms with the compact-open topology becomes a compact right $\Lambda_G(k)$ -module by $(\varphi g)(m) = \varphi(m)g$, $\varphi \in \text{Hom}_{\Lambda_G(k)}(M, N)$, $g \in G$, $m \in M$. The continuous cohomology group $H^1(G, M)$ has a natural structure as a right $\Lambda_G(k)$ -module which is induced by the action on the group of 1-cocycles: Let $a: G \rightarrow M$ be a 1-cocycle and $g \in G$, then $(a \cdot g)(x) = a(x)g$ for $x \in G$.

Lemma 1.3 *Let G be a pro- p group.*

(i) *There is a natural Λ_G -isomorphism*

$$H^1(G, \mathbb{F}_p[[G]]) \cong \varprojlim_U H^1(G, \mathbb{F}_p[G/U]),$$

where U runs through the normal open subgroups of G .

(ii) *Let H be an open subgroup of G . Then there is a natural Λ_H -isomorphism*

$$H^1(G, \mathbb{F}_p[[G]]) \xrightarrow{sh} H^1(H, \mathbb{F}_p[[H]]).$$

(iii) *If G is finitely generated, then there are natural Λ_G -isomorphisms*

$$H^1(G, \mathbb{F}_p[[G]]) \cong (\varinjlim_{U, \text{Ver}_G^U} U/U^*)^\vee \cong D_1(G, \mathbb{F}_p)^\vee,$$

where the limit is taken over the transfer maps.

Proof: Using [12] (2.7.6) and Shapiro's lemma [12] (1.6.4), we have

$$\begin{aligned} H^1(G, \mathbb{F}_p[[G]]) &\cong \varprojlim_U H^1(G, \mathbb{F}_p[G/U]) \cong \varprojlim_U H^1(U, \mathbb{F}_p), \\ H^1(G, \mathbb{F}_p[[G]]) &= H^1(G, \text{Ind}_H^G \mathbb{F}_p[[H]]) \xrightarrow{sh} H^1(H, \mathbb{F}_p[[H]]). \end{aligned}$$

If G is finitely generated, then the groups $H^1(G, \mathbb{F}_p[G/U])$ are finite and the Λ_G -module $H^1(G, \mathbb{F}_p[[G]])$ is compact. Pontryagin-duality gives

$$H^1(G, \mathbb{F}_p[[G]]) \cong (\varinjlim_U H^1(U, \mathbb{F}_p)^\vee)^\vee \cong (\varinjlim_U U/U^*)^\vee \cong D_1(G, \mathbb{F}_p)^\vee.$$

□

Let G be an infinite pro- p group. Then $\text{Hom}_{\Lambda_G}(\mathbb{F}_p, \Lambda_G) = 0$. Since

$$\text{Ext}_{\Lambda_G}^1(\mathbb{F}_p, \Lambda_G) = H^1(G, \Lambda_G) \quad \text{and} \quad \text{Hom}_{\Lambda_G}(\Lambda_G, \Lambda_G) = \Lambda_G,$$

see [12] (5.2.14), the exact sequence

$$0 \longrightarrow I_G \longrightarrow \Lambda_G \longrightarrow \mathbb{F}_p \longrightarrow 0$$

yields the exact sequence

$$(*) \quad 0 \longrightarrow \Lambda_G \xrightarrow{\phi_G} \text{Hom}_{\Lambda_G}(I_G, \Lambda_G) \longrightarrow H^1(G, \Lambda_G) \longrightarrow 0,$$

where $\phi_G(\lambda) : x \mapsto x\lambda$.

Proposition 1.4 *Let G be a pro- p group.*

- (i) *If I_G is decomposable, then G is infinite and $h^1(G) \neq 0$.*
- (ii) *If $h^1(G) \neq 0$, then there exists an open subgroup H of G such that I_H has a direct summand isomorphic to Λ_H .*
- (iii) *If $I_G \cong \Lambda_G$, then $G \cong \mathbb{Z}_p$ and $h^1(G) = 1$.*
- (iv) *If $0 < h^1(G) < \infty$, then $h^1(G) = 1$ and there exists an open subgroup H of G isomorphic to \mathbb{Z}_p .*

Proof: (i) Let $I_G = M_1 \oplus M_2$ be a non-trivial decomposition of I_G . If G is finite, then $(M_1)^G \oplus (M_2)^G = (I_G)^G$ and $\dim_{\mathbb{F}_p}(I_G)^G = 1$ implies that $M_1 = 0$ or $M_2 = 0$. Thus G has to be infinite. We consider the exact sequence

$$0 \longrightarrow \Lambda_G \longrightarrow \text{Hom}_{\Lambda_G}(M_1, \Lambda_G) \oplus \text{Hom}_{\Lambda_G}(M_2, \Lambda_G) \longrightarrow H^1(G, \Lambda_G) \longrightarrow 0.$$

Since the groups $\text{Hom}_{\Lambda_G}(M_j, \Lambda_G)$, $j = 1, 2$, are non-zero and Λ_G is indecomposable, assertion (i) follows.

(ii) Since $H^1(G, \Lambda_G) = \varprojlim_U H^1(U, \mathbb{F}_p) \neq 0$, see (1.3)(i), and since the diagram

$$\begin{array}{ccc} H^1(G, \Lambda_G) & \longrightarrow & H^1(G, \mathbb{F}_p) \\ sh \downarrow \sim & & \uparrow cor \\ H^1(U, \Lambda_U) & \longrightarrow & H^1(U, \mathbb{F}_p) \end{array}$$

commutes, see [12] (1.6.5), there exists an open subgroup H of G such that the canonical map $H^1(H, \mathbb{F}_p[[H]]) \rightarrow H^1(H, \mathbb{F}_p)$ induce by the augmentation map $\mathbb{F}_p[[H]] \rightarrow \mathbb{F}_p$ is not zero. Now the commutative and exact diagram

$$\begin{array}{ccccccc} & & \text{Hom}_{\Lambda_H}(I_H, \mathbb{F}_p) & \xrightarrow{\sim} & H^1(H, \mathbb{F}_p) & & \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \Lambda_H & \longrightarrow & \text{Hom}_{\Lambda_H}(I_H, \Lambda_H) & \longrightarrow & H^1(H, \Lambda_H) \longrightarrow 0 \end{array}$$

shows that there exists a Λ_H -homomorphism $I_H \rightarrow \Lambda_H$ which is surjective as its image is not contained in I_H . It follows that $I_H \cong \Lambda_H \oplus M$ with some Λ_H -module M (possibly trivial).

(iii) The exact sequence $0 \rightarrow \Lambda_G \rightarrow \Lambda_G \rightarrow \mathbb{F}_p \rightarrow 0$ shows that the projective dimension of \mathbb{F}_p as Λ_G -module is equal to 1, hence $\text{cd}_p G = 1$, see [12] (5.2.13). Since $I_G \cong (\Lambda_G)^r$ for a free pro- p group G of rank r , [12] (5.6.3), (5.6.4), we obtain $G \cong \mathbb{Z}_p$, and so $h^1(G) = 1$.

(iv) Since $0 < h^1(G) < \infty$, G has to be infinite by Shapiro's lemma. Using (ii), we see that there exists an open subgroup H of G such that I_H has a direct summand isomorphic to Λ_H , i.e. $I_H \cong \Lambda_H \oplus M$ for some Λ_H -module M . We get the commutative and exact diagram

$$\begin{array}{ccccccc} & & \text{Hom}_{\Lambda_H}(M, \Lambda_H) & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & \Lambda_H & \xrightarrow{\phi_H} & \text{Hom}_{\Lambda_H}(I_H, \Lambda_H) & \longrightarrow & H^1(H, \Lambda_H) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \\ & & \Lambda_H & \xrightarrow{\varepsilon} & \text{Hom}_{\Lambda_H}(\Lambda_H, \Lambda_H) & & \end{array}$$

The image of ϕ_H consists of homomorphisms $I_H \rightarrow \Lambda_H$ given by right-multiplication with an element $\lambda \in \Lambda_H$. Since the Λ_H -annulator of Λ_H is zero, the map ε is injective. It follows that $\text{Hom}_{\Lambda_H}(M, \Lambda_H)$ injects into $H^1(H, \Lambda_H) \cong H^1(G, \Lambda_G)$. Hence this module is finite and so fixed by an open subgroup H_0 of H . Therefore the map $\iota \in \text{Hom}_{\Lambda_H}(M, \Lambda_H)$, $\iota: M \hookrightarrow I_H \hookrightarrow \Lambda_H$, has an image in $(\Lambda_H)^{H_0} = 0$, hence $M = 0$. Thus $I_H \cong \Lambda_H$, and so $H \cong \mathbb{Z}_p$ and $1 = h^1(H) = h^1(G)$ by (iii). This proves (iv). \square

Proof of Theorem 1: Obviously $h^0(G) \leq 1$ and $h^0(G) = 1$ if and only if G is finite. Using Shapiro's lemma, we obtain assertion (ii).

Assume that $0 < h^1(G) < \infty$. Then G has to be infinite by Shapiro's lemma and $e(G) = h^1(G) + 1$. From (1.4)(iv) it follows that $h_1(G) = 1$ and there exists an open subgroup H of G isomorphic to \mathbb{Z}_p , i.e. the assertions (i) and (iii) hold.

If G is torsion-free, then a theorem of Serre, see [15], implies that $\text{cd}_p G = 1$, and so $G \cong \mathbb{Z}_p$. \square

Lemma 1.5 *Let G and H be pro- p groups and k a finite field of characteristic p . Let M be a left compact $\Lambda_H(k)$ -module, L a compact $(\Lambda_G(k), \Lambda_H(k))$ -bimodule and N a left $\Lambda_G(k)$ -module. Assume either that N is compact and M and L are finitely generated or that N is discrete.*

(i) *There is a canonical isomorphism*

$$S : \text{Hom}_{\Lambda_H(k)}(M, \text{Hom}_{\Lambda_G(k)}(L, N)) \xrightarrow{\sim} \text{Hom}_{\Lambda_G(k)}(L \hat{\otimes}_{\Lambda_H(k)} M, N),$$

such that $(S\varphi)(l \hat{\otimes} m) = \varphi(m)(l)$. This morphism induces a natural equivalence of functors.

(ii) *Let H be a closed subgroup of G . Considering $\Lambda_G(k)$ as $(\Lambda_G(k), \Lambda_H(k))$ -bimodule, there are canonical isomorphisms*

$$\text{Ext}_{\Lambda_H(k)}^i(M, \text{Res}_H N) \xrightarrow{\sim} \text{Ext}_{\Lambda_G(k)}^i(\Lambda_G(k) \hat{\otimes}_{\Lambda_H(k)} M, N), \quad i \geq 0,$$

where $\text{Res}_H N$ denotes the $\Lambda_H(k)$ -module N obtained by restriction of scalars.

Proof: If A and $B = \varprojlim_i B_i$ are compact $\Lambda_G(k)$ -modules (B_i finite $\Lambda_G(k)$ -modules), where A is finitely generated, then

$$\text{Hom}_{\Lambda_G(k)}(A, B) = \varprojlim_i \text{Hom}_{\Lambda_G(k)}(A, B_i)$$

is a compact $\Lambda_G(k)$ -module.

Now assertion (i) is the topological analog of [4] chap.II (5.2) or [6] (2.19): see [3](2.4), where N is discrete, and take the projective limit if N is compact.

In order to prove (ii), take $L = \Lambda_G(k)$ and observe that $\text{Hom}_{\Lambda_G(k)}(\Lambda_G(k), N)$ and $\text{Res}_H N$ are isomorphic as left $\Lambda_H(k)$ -modules. Then we obtain from (i) the assertion for $i = 0$. Since $\Lambda_G(k)$ is $\Lambda_H(k)$ -projective, the functor $\Lambda_G(k) \hat{\otimes}_{\Lambda_H(k)} -$ is exact, and so, taking a $\Lambda_H(k)$ -projective resolution of M , we obtain the desired isomorphisms for all $i \geq 0$, see also [4] VI. Proposition (4.1.3). \square

Let G be a pro- p group and let H be a closed subgroup. If I is a left ideal of Λ_H , then $\Lambda_G \hat{\otimes}_{\Lambda_H} I \cong \Lambda_G I$, see [17] (4.3): Applying the exact functor $\Lambda_G \hat{\otimes}_{\Lambda_H} -$

to the exact sequence $0 \rightarrow I \rightarrow \Lambda_H$ gives $0 \rightarrow \Lambda_G \hat{\otimes}_{\Lambda_H} I \rightarrow \Lambda_G$ with image $\Lambda_G I$. In particular,

$$J_H := \Lambda_G \hat{\otimes}_{\Lambda_H} I_H = \text{Ind}_H^G I_H$$

is the left ideal of Λ_G generated by I_H . The Frobenius reciprocity (1.5)(ii) gives the natural isomorphisms

$$\text{Ext}_{\Lambda_H}^i(I_H, \text{Res}_H N) \cong \text{Ext}_{\Lambda_G}^i(J_H, N)$$

for $i \geq 0$, where N is a discrete left Λ_G -module or N is compact and H is finitely generated.

The following result is a pro- p analog of a theorem for abstract groups, see [5] theorem (4.7).

Proposition 1.6 *Let G be a pro- p group, and let H_j , $1 \leq j \leq n$, be closed subgroups. Then the following assertions are equivalent*

- (i) $G = H_1 * \cdots * H_n$.
- (ii) $I_G = J_{H_1} \oplus \cdots \oplus J_{H_n}$.

Proof: By [12] (4.1.5) we know that (i) is equivalent to

$$(iii) \quad H^i(G, \mathbb{F}_p) \xrightarrow[\sim]{res} \bigoplus_{j=1}^n H^i(H_j, \mathbb{F}_p), \quad i = 1, 2.$$

From the exact sequence $0 \rightarrow I_G \rightarrow \Lambda_G \rightarrow \mathbb{F}_p \rightarrow 0$ we obtain the isomorphisms

$$\text{Ext}_{\Lambda_G}^i(I_G, \mathbb{F}_p) \xrightarrow{\sim} \text{Ext}_{\Lambda_G}^{i+1}(\mathbb{F}_p, \mathbb{F}_p) = H^{i+1}(G, \mathbb{F}_p), \quad i = 0, 1.$$

Since $\text{Ext}_{\Lambda_G}^i(J_{H_j}, \mathbb{F}_p) \cong \text{Ext}_{\Lambda_{H_j}}^i(I_{H_j}, \mathbb{F}_p)$, assertion (iii) is equivalent to

$$(iv) \quad \text{Ext}_{\Lambda_G}^i(I_G, \mathbb{F}_p) \xrightarrow[\sim]{can} \bigoplus_{j=1}^n \text{Ext}_{\Lambda_G}^i(J_{H_j}, \mathbb{F}_p), \quad i = 0, 1.$$

Therefore (ii) implies (i). Conversely, from the exact sequence

$$0 \longrightarrow M \longrightarrow \bigoplus_{j=1}^n J_{H_j} \longrightarrow I_G \longrightarrow 0,$$

where M is defined as kernel of the natural surjection on the right, we obtain the exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\Lambda_G}(I_G, \mathbb{F}_p) &\rightarrow \bigoplus_{j=1}^n \text{Hom}_{\Lambda_G}(J_{H_j}, \mathbb{F}_p) \rightarrow \text{Hom}_{\Lambda_G}(M, \mathbb{F}_p) \\ &\rightarrow \text{Ext}_{\Lambda_G}^1(I_G, \mathbb{F}_p) \rightarrow \bigoplus_{j=1}^n \text{Ext}_{\Lambda_G}^1(J_{H_j}, \mathbb{F}_p). \end{aligned}$$

Using (iv), we obtain $\text{Hom}_{\Lambda_G}(M, \mathbb{F}_p) = 0$, hence $M = 0$. This completes the proof of the proposition. \square

The following corollary can also be obtained from theorem (3.4) in [13].

Corollary 1.7 *Let G be a finitely generated pro- p duality group of dimension $n \geq 2$. Then G is freely indecomposable.*

Proof: By [12] (3.4.6) we have $D_1(G, \mathbb{F}_p) = 0$, if G is a duality group of dimension $n \geq 2$. Using lemma (1.3)(iii), it follows that $h^1(G) = 0$. Hence proposition (1.6) and proposition (1.4)(i) gives the result. \square

In order to establish a decomposition of the form $I_G = J_{H_1} \oplus \cdots \oplus J_{H_n}$ we will need the following lemmata.

Lemma 1.8 *Let U be a closed subgroup of the pro- p group G and let k be a finite field of characteristic p . Then*

- (i) *$\text{Res}_U I_G(k) = I_U(k) \oplus P$, where P is a free $\Lambda_U(k)$ -module. If U is open and normal in G of index d and if $\{\eta_1, \dots, \eta_{d-1}\} \subseteq I_G(k)$ is an arbitrary pre-image of a k -basis of $I_{G/U}(k)$ under the canonical surjection $I_G(k) \rightarrow I_{G/U}(k)$, then*

$$\text{Res}_U I_G(k) = I_U(k) \oplus \bigoplus_{i=1}^{d-1} \Lambda_U(k) \eta_i.$$

- (ii) *If M is a finitely generated $\Lambda_U(k)$ -module, then*

$$M | \text{Res}_U \text{Ind}_U^G M.$$

Proof: Consider the commutative and exact diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Res}_U I_G(k) & \longrightarrow & \text{Res}_U \Lambda_G(k) & \longrightarrow & k \longrightarrow 0 \\ & & \uparrow & & \uparrow \varphi & & \parallel \\ 0 & \longrightarrow & I_U(k) & \longrightarrow & \Lambda_U(k) & \longrightarrow & k \longrightarrow 0, \end{array}$$

where $\varphi(\Lambda_U(k))$ is some free summand (of rank 1) of the free $\Lambda_U(k)$ -module $\text{Res}_U \Lambda_G(k)$. It follows that $\text{Res}_U I_G(k)$ surjects onto the free $\Lambda_U(k)$ -module $P = \text{Res}_U \Lambda_G(k) / \varphi(\Lambda_U(k))$, showing the first statement of (i). Since $\text{Res}_U \Lambda_G(k) = \bigoplus_{i=0}^{d-1} \Lambda_U(k) \eta_i$, $\eta_0 = 1$, the second is also obvious.

In order to prove (ii), let N be an open normal subgroup of G . By the double coset formula we have

$$\text{Res}_{UN/N} \text{Ind}_{UN/N}^{G/N} M_{U \cap N} = \bigoplus_{a \in \mathcal{R}} \text{Ind}_{UN/N \cap (UN/N)^a}^{UN/N} a \text{Res}_{UN/N \cap (UN/N)^a} M_{U \cap N},$$

where \mathcal{R} is a system of representatives of the double coset decomposition $G/N = \bigcup_a (UN/N)a(UN/N)$. In particular, $M_{U \cap N} | \text{Res}_{UN/N} \text{Ind}_{UN/N}^{G/N} M_{U \cap N}$. Passing to the limit over all open normal subgroups of G gives the result. \square

Recall that a $\Lambda_G(k)$ -module M is called G - H -projective, where H is a closed subgroup of the profinite group G , iff ever a $\Lambda_G(k)$ -epimorphism $B \twoheadrightarrow M$ splits as a $\Lambda_H(k)$ -module homomorphism, then it splits as a $\Lambda_G(k)$ -module homomorphism, see [10] (3.1),(3.2).

Lemma 1.9 *Let G be a finitely generated pro- p group and H a closed subgroup of G . Let M be a finitely generated Λ_G -module and $k|\mathbb{F}_p$ a finite extension of fields. Then*

- (i) M is G - H -projective $\Leftrightarrow k\hat{\otimes}_{\mathbb{F}_p} M$ is G - H -projective.
- (ii) Assume in addition that $M|I_G$ and M is indecomposable and not isomorphic to Λ_G . Then

$$M| \text{Ind}_H^G I_H \Leftrightarrow k\hat{\otimes}_{\mathbb{F}_p} M| \text{Ind}_H^G I_H(k).$$

Proof: (i) If A is a Λ_G -module, then we put $\bar{A} = k\hat{\otimes}_{\mathbb{F}_p} A$. Let \bar{M} be G - H -projective and let

$$(*) \quad 0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$$

be an exact sequence of Λ_G -modules which is H -split. Applying the exact functor $k\hat{\otimes}_{\mathbb{F}_p} -$, we obtain the exact sequence $0 \rightarrow \bar{A} \rightarrow \bar{B} \rightarrow \bar{M} \rightarrow 0$ of $\Lambda_G(k)$ -modules which is H -split, hence G -split. Therefore we obtain for every open normal subgroup N of G a split exact sequence

$$0 \longrightarrow \bar{A}_N \longrightarrow \bar{B}_N \longrightarrow \bar{M}_N \longrightarrow 0$$

of $k[G/N]$ -modules. It follows that the sequences $0 \rightarrow A_N \rightarrow B_N \rightarrow M_N \rightarrow 0$ are exact and these sequences split since

$$k \otimes_{\mathbb{F}_p} \text{Ext}_{\mathbb{F}_p[G/H]}(M_N, A_N) \xrightarrow{\sim} \text{Ext}_{k[G/H]}(\bar{M}_N, \bar{A}_N),$$

see [6] (8.16). Thus the sequence $(*)$ is G -split.

Conversely, if M is G - H -projective, then $M| \text{Ind}_H^G \text{Res}_H M$, [10] (3.2), and so $k\hat{\otimes}_{\mathbb{F}_p} M| \text{Ind}_H^G \text{Res}_H(k\hat{\otimes}_{\mathbb{F}_p} M)$. Again by [10] (3.2), it follows that $k\hat{\otimes}_{\mathbb{F}_p} M$ is G - H -projective.

(ii) In order to prove the non-trivial implication, let $k\hat{\otimes}_{\mathbb{F}_p} M| \text{Ind}_H^G I_H(k)$. Then $k\hat{\otimes}_{\mathbb{F}_p} M$ is G - H -projective, and so M is G - H -projective. Using (1.8)(i), we have $\text{Res}_H M| I_H \oplus P_H$, where P_H is a free Λ_H -module, and, using [10] (3.2), we obtain

$$M| \text{Ind}_H^G \text{Res}_H M| \text{Ind}_H^G I_H \oplus P_G,$$

where P_G is a free Λ_G -module. Since M is indecomposable and not free, it follows that $M| \text{Ind}_H^G I_H$, see (1.2). \square

Lemma 1.10 *Let G be a finitely generated pro- p group. Assume that there is a decomposition*

$$I_G = P \oplus M$$

of left Λ_G -modules where P is free of rank r . Then there exists a free pro- p subgroup F of G of rank r such that

$$I_G = \text{Ind}_F^G I_F \oplus M.$$

Proof: Let $n = \dim_{\mathbb{F}_p} G/G^*$ and

$$G/G^* = \langle \bar{x}_1, \dots, \bar{x}_r \rangle \oplus \langle \bar{x}_{r+1}, \dots, \bar{x}_n \rangle$$

be the decomposition of G/G^* corresponding to

$$G/G^* \xrightarrow{\text{can}} I_G/I_G^2 = P_G \oplus M_G,$$

where can is induced by the map $\sigma \mapsto \sigma - 1$, and so $P_G = \langle \bar{x}_i - 1, i = 1, \dots, r \rangle$. Let $x_1, \dots, x_n \in G$ be arbitrary pre-images of the \bar{x}_i 's, and let $F = \langle x_1, \dots, x_r \rangle$. Let F_n be a free pro- p group with basis $\{y_1, \dots, y_n\}$ and let

$$1 \longrightarrow R \longrightarrow F_n \xrightarrow{\pi} G \longrightarrow 1, \quad \pi(y_i) = x_i, \quad i = 1, \dots, n,$$

a be minimal representation of G . If F_r denotes the free subgroup of F_n generated by $\{y_1, \dots, y_r\}$, then F_r is mapped onto F . Since

$$I_{F_n} = \bigoplus_{i=1}^n \Lambda_{F_n}(y_i - 1) \quad \text{and} \quad I_{F_n}/I_R I_{F_n} = \bigoplus_{i=1}^n \Lambda_G(y_i - 1),$$

see [12] (5.6.4), (5.6.6), we obtain a commutative diagram

$$\begin{array}{ccccc} & & \xrightarrow{\varphi} & & \\ \bigoplus_{i=1}^n \Lambda_G(y_i - 1) & \xrightarrow{\tilde{\pi}} & I_G & \xrightarrow{\xi} & P \\ & \uparrow \iota & \uparrow & \nearrow \psi & \\ \bigoplus_{i=1}^r \Lambda_G(y_i - 1) & \xrightarrow{\tilde{\pi}} & X & & \end{array}$$

where $\tilde{\pi}$ is induced by π , X is the image of $(\text{Ind}_{F_r}^{F_n} I_{F_r})_R = \bigoplus_{i=1}^r \Lambda_G(y_i - 1)$ under $\tilde{\pi}$, ξ is the surjection of I_G onto the free factor given by assumption and φ resp. ψ its composition with $\tilde{\pi}$ resp. the inclusion. The map $\iota\varphi = \psi\tilde{\pi}$ is surjective, and so ψ and $\tilde{\pi}$ restricted to $\bigoplus_{i=1}^r \Lambda_G(y_i - 1)$ are bijective. It follows that $X = \text{Ind}_F^G I_F$ is a free Λ_G -module of rank r , and so F is free of rank r , and $I_G = X \oplus M$. \square

Lemma 1.11 *Let G be a pro- p group and k a finite field of characteristic p . Let*

$$I_G(k) = M \oplus R$$

be a non-trivial decomposition of $I_G(k)$ into left $\Lambda_G(k)$ -modules, where M is finitely generated, indecomposable and not free. Then there exists a proper open normal subgroup E of G such that

$$M | \text{Ind}_E^G I_E(k).$$

Proof: Let \tilde{E} be a closed subgroup of G such that $(\tilde{E}G^*/G^*) \hat{\otimes}_{\mathbb{F}_p} k$ corresponds to M_G in the decomposition

$$G/G^* \hat{\otimes}_{\mathbb{F}_p} k \xrightarrow{\text{can}} I_G(k)/I_G(k)^2 = M_G \oplus R_G.$$

Let $E \subseteq G$ be a subgroup of index p containing \tilde{E} (which exists, as the decomposition of $I_G(k)$ is non-trivial). Then the map

$$\tilde{\varphi}: R \rightarrow R_G \hookrightarrow I_G(k)/I_G(k)^2 \simeq (G/G^*) \hat{\otimes}_{\mathbb{F}_p} k \rightarrow (G/E) \hat{\otimes}_{\mathbb{F}_p} k$$

is surjective. By Nakayama's lemma it follows that the map

$$\varphi: R \hookrightarrow I_G(k) \rightarrow I_{G/E}(k)$$

is surjective, since $R \xrightarrow{\varphi} I_{G/E}(k) \rightarrow I_{G/E}(k)/I_{G/E}(k)^2 \cong (G/E) \hat{\otimes}_{\mathbb{F}_p} k$ is the surjective map $\tilde{\varphi}$. Thus we have the canonical exact and commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ind}_E^G I_E(k) & \longrightarrow & I_G(k) & \longrightarrow & I_{G/E}(k) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & R' & \longrightarrow & R & \xrightarrow{\varphi} & I_{G/E}(k) \longrightarrow 0, \end{array}$$

where $R' = R \cap \text{Ind}_E^G I_E(k)$ is the kernel of φ . Let $\psi: I_G(k) \rightarrow I_{G/E}(k)$ be the map which is equal to φ when restricted to R and zero when restricted to M . We consider the commutative and exact diagram

$$(*) \quad \begin{array}{ccccccc} & & & k & \xlongequal{\quad} & k & \\ & & & \uparrow & & \uparrow & \\ 0 & \longrightarrow & M \oplus R' & \longrightarrow & \Lambda_G(k) & \xrightarrow{\tilde{\psi}} & X \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & M \oplus R' & \longrightarrow & I_G(k) & \xrightarrow{\psi} & I_{G/E}(k) \longrightarrow 0, \end{array}$$

where X is defined as the quotient $\Lambda_G(k)/(M \oplus R')$ and $\tilde{\psi}$ is the projection.

Claim: $X \cong \Lambda_{G/E}(k)$.

Proof: Let $\{\eta_1, \dots, \eta_{p-1}\} \subseteq R$ be a pre-image with respect to φ of a k -basis of $I_{G/E}(k)$. Using (1.8)(i), we get $\text{Res}_E I_G(k) = I_E(k) \oplus \bigoplus_{i=1}^{p-1} \Lambda_E(k) \eta_i$. Since $\bigoplus_{i=1}^{p-1} \Lambda_E(k) \eta_i \subseteq \text{Res}_E R$ is a direct summand of $\text{Res}_E I_G(k)$, we obtain

$$\text{Res}_E R = \tilde{R} \oplus \bigoplus_{i=1}^{p-1} \Lambda_E(k) \eta_i, \quad \text{Res}_E R' = \tilde{R} \oplus \bigoplus_{i=1}^{p-1} I_E(k) \eta_i,$$

where $\tilde{R} = I_E(k) \cap \text{Res}_E R$. It follows that

$$\begin{aligned} \text{Res}_E M \oplus \tilde{R} \oplus \bigoplus_{i=1}^{p-1} \Lambda_E(k) \eta_i &= \text{Res}_E M \oplus \text{Res}_E R \\ &= \text{Res}_E I_G(k) \\ &= I_E(k) \oplus \bigoplus_{i=1}^{p-1} \Lambda_E(k) \eta_i, \end{aligned}$$

and so $\text{Res}_E M \oplus \tilde{R} \cong I_E(k)$. Applying the restriction functor to the diagram (*), we obtain the commutative and exact diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Res}_E M \oplus \tilde{R} \oplus \bigoplus_{i=1}^{p-1} \Lambda_E(k) \eta_i & \longrightarrow & \Lambda_E(k) \oplus \bigoplus_{i=1}^{p-1} \Lambda_E(k) \eta_i & \longrightarrow & \text{Res}_E X \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \text{Res}_E M \oplus \tilde{R} \oplus \bigoplus_{i=1}^{p-1} I_E(k) \eta_i & \longrightarrow & I_E(k) \oplus \bigoplus_{i=1}^{p-1} \Lambda_E(k) \eta_i & \longrightarrow & k^{p-1} \longrightarrow 0, \end{array}$$

and, dividing out the term $\bigoplus_{i=1}^{p-1} I_E(k)$, the commutative and exact diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Res}_E M \oplus \tilde{R} & \longrightarrow & \Lambda_E(k) \oplus \bigoplus_{i=1}^{p-1} k & \longrightarrow & \text{Res}_E X \longrightarrow 0 \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \text{Res}_E M \oplus \tilde{R} & \longrightarrow & I_E(k) \oplus \bigoplus_{i=1}^{p-1} k & \xrightleftharpoons{\quad} & k^{p-1} \longrightarrow 0. \end{array}$$

Since $\text{Res}_E M \oplus \tilde{R} \cong I_E(k)$, we obtain $\text{Res}_E X \cong k^p$, and so $\text{Res}_E X$ is a trivial E -module, i.e. X is a G/E -module and we obtain a surjection $\Lambda_{G/E}(k) \twoheadrightarrow X$ which has to be an isomorphism. This proves the claim.

Thus we have two exact sequences of $\Lambda_G(k)$ -modules

$$0 \longrightarrow M \oplus R' \longrightarrow \Lambda_G(k) \longrightarrow \Lambda_{G/E}(k) \longrightarrow 0$$

$$0 \longrightarrow \text{Ind}_E^G I_E(k) \longrightarrow \Lambda_G(k) \longrightarrow \Lambda_{G/E}(k) \longrightarrow 0.$$

The lemma of Schanuel implies that $M \oplus R' \oplus \Lambda_G(k) \cong \text{Ind}_E^G I_E(k) \oplus \Lambda_G(k)$, see [6] (2.24), thus $M | \text{Ind}_E^G I_E(k)$ by (1.2). \square

Lemma 1.12 *Let G be a pro- p group and H a closed subgroup. Let*

$$M | \text{Ind}_H^G I_H$$

be a finitely generated Λ_G -module. Assume that H is not finitely generated. Then there exists a proper open subgroup E of H such that

$$M | \text{Ind}_E^G I_E.$$

Proof: We may assume that M is contained in $\text{Ind}_H^G I_H$. Let N be an open normal subgroup of G , E a proper open normal subgroup of H containing H^* and

$$\varphi_N^E: M_N \hookrightarrow (\text{Ind}_H^G I_H)_N \twoheadrightarrow (\text{Ind}_H^G I_{H/E})_N = \text{Ind}_{HN/N}^{G/N}(I_{H/E})_{N \cap H}.$$

Then the sets

$$\mathcal{B}_N = \{H^* \subseteq E \subsetneq H \mid E \text{ open and normal in } H, \varphi_N^E = 0\},$$

where N runs through the open normal subgroups of G , form a projective system. Furthermore, $\mathcal{B}_N = \varprojlim_{H'} \mathcal{B}_{N,H'}$, where $H' \subsetneq H$ is open and normal, $H^* \subseteq H'$, and $\mathcal{B}_{N,H'} = \{H' \subseteq E \subsetneq H \mid E \text{ open and normal in } H, \varphi_N^E = 0\}$; the transition maps are given by $\mathcal{B}_{N,H''} \rightarrow \mathcal{B}_{N,H'}$, $E \mapsto EH'$, if $H'' \subseteq H'$. Since the sets $\mathcal{B}_{N,H'}$ are finite, the set \mathcal{B}_N is compact. We will show that these sets are not empty. Then $\varprojlim_N \mathcal{B}_N$ is not empty, i.e. there exists a proper open normal subgroup E of H such that the map

$$\varphi^E: M \hookrightarrow \text{Ind}_H^G I_H \twoheadrightarrow \text{Ind}_H^G I_{H/E}$$

is zero. It follows that $M \subseteq \text{Ind}_E^G I_E$, hence $M | \text{Ind}_E^G I_E$.

Claim: $\mathcal{B}_N \neq \emptyset$.

Since M_N has finite \mathbb{F}_p -dimension, the image of the map

$$M_N \hookrightarrow (\text{Ind}_H^G I_H)_N \twoheadrightarrow (\text{Ind}_H^G I_{H/H^*})_N = \text{Ind}_{HN/N}^{G/N}(I_{H/H^*})_{N \cap H}$$

is contained in $\text{Ind}_{HN/N}^{G/N} A$, where $A \subseteq (I_{H/H^*})_{N \cap H}$ is a $H/N \cap H$ -module and finite dimensional as \mathbb{F}_p -vector space. Let $\tilde{N} = (N \cap H)H^*$. Considering the commutative and exact diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{N}/\tilde{N}^* & \longrightarrow & (I_{H/H^*})_{\tilde{N}} & \longrightarrow & I_{H/\tilde{N}} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & A_0 & \longrightarrow & A & \longrightarrow & A/A_0 \longrightarrow 0. \end{array}$$

where $A_0 = A \cap \tilde{N}/\tilde{N}^*$. Since \tilde{N}/\tilde{N}^* has infinite \mathbb{F}_p -dimension and A_0 is finite dimensional, there exists proper subspace $0 \neq \tilde{E} \subsetneq H/H^*$ such that $\tilde{E} \subseteq \tilde{N}/\tilde{N}^*$

and $\tilde{E} \cap A_0 = 0$. Let E be a proper open normal subgroup of H containing H^* such that $\tilde{E} \oplus E/H^* = H/H^*$, and so $H/E \cong \tilde{E}$. It follows that

$$(I_{H/E})_{\tilde{N}} = I_{H/E}/I_{\tilde{N}}I_{H/E} = I_{H/E}/(I_{H/E})^2 = H/E \quad \text{and} \quad I_{H/\tilde{N}E} = 0.$$

Thus the map $A \hookrightarrow (I_{H/H^*})_{\tilde{N}} \twoheadrightarrow (I_{H/E})_{\tilde{N}}$ is zero, and so $\varphi_N^E = 0$. This proves the claim and so the lemma. \square

Proposition 1.13 *Let G be a finitely generated pro- p group. Let*

$$I_G = M \oplus R$$

be a decomposition of the augmentation ideal I_G into left Λ_G -modules, where M is indecomposable and not isomorphic to Λ_G . Then there exists a finitely generated closed subgroup H of G such that

$$M \cong \text{Ind}_H^G I_H.$$

Proof: We may assume that the decomposition of I_G is non-trivial (otherwise take $H = G$). We consider the set

$$\mathcal{M} = \{U \mid U \text{ a closed subgroup of } G \text{ such that } M \mid \text{Ind}_U^G I_U\},$$

which is partially ordered when ordered by inclusion and non-empty. Furthermore, if $\{U_1 \supseteq U_2 \supseteq \dots\}$ is a chain in \mathcal{M} , then $U = \bigcap_i U_i$ is a lower bound in \mathcal{M} ; indeed, from [10] (3.7) and (4.2) it follows that $M \mid \text{Ind}_U^G \text{Res}_U M$. Since $M \mid I_G$ and so $\text{Res}_U M \mid \text{Res}_U I_G$, we obtain, using (1.8)(i),

$$\text{Res}_U M \mid (I_U \oplus P_0), \text{ where } P_0 \text{ is } \Lambda_U\text{-free,}$$

hence

$$M \mid (\text{Ind}_U^G I_U \oplus P), \text{ where } P \text{ is } \Lambda_G\text{-free.}$$

Using (1.2), it follows that $M \mid \text{Ind}_U^G I_U$, since $M \not\cong \Lambda_G$. Now Zorn's lemma implies that \mathcal{M} has a minimal element H and we have $M \mid \text{Ind}_H^G I_H$.

The group H is finitely generated, since otherwise, by (1.12), there would exist a proper open subgroup H_0 of H such that $M \mid \text{Ind}_{H_0}^G I_{H_0}$ which contradicts the minimality of H .

Furthermore, $I_H(k)$ is indecomposable for every finite extension $k \mid \mathbb{F}_p$. Indeed, suppose the contrary, i.e. $I_H(k) = A \oplus B$ is a non-trivial decomposition and $k \otimes_{\mathbb{F}_p} M \mid \text{Ind}_H^G A$, where A is an indecomposable $\Lambda_H(k)$ -module not isomorphic to $\Lambda_H(k)$. From (1.11) it follows that $A \mid \text{Ind}_E^H I_E(k)$ where E is a proper open subgroup of H . Thus $k \otimes_{\mathbb{F}_p} M \mid \text{Ind}_E^G I_E(k)$. Using (1.9)(ii), we get a contradiction to the minimality of H .

Now Green's indecomposability theorem for pro- p groups says that $\text{Ind}_H^G I_H$ is indecomposable, see [10](6.7), thus $M \cong \text{Ind}_H^G I_H$. \square

Theorem 1.14 *Let G be a finitely generated pro- p group and let*

$$I_G = M_1 \oplus \cdots \oplus M_s$$

be a decomposition into indecomposable left Λ_G -modules M_i . Then there exist freely indecomposable closed subgroups H_i of G such that $\text{Ind}_{H_i}^G I_{H_i} \cong M_i$, $i = 1, \dots, s$, and

$$G = H_1 * \cdots * H_s.$$

In particular, G is freely indecomposable if and only if I_G is indecomposable.

Proof: Let $r = \#\{i | M_i \cong \Lambda_G\}$ and $t = s - r$. By (1.13) we obtain (after re-numbering)

$$I_G = M \oplus P,$$

$$M \cong \text{Ind}_{H_1}^G I_{H_1} \oplus \cdots \oplus \text{Ind}_{H_t}^G I_{H_t}$$

where H_i , $i = 1, \dots, t$, are closed subgroups of G and P is a free Λ_G -module of rank r and M has no free summand. If $H = \langle H_1, \dots, H_t \rangle \subseteq G$, then

$$I_G = \text{Ind}_H^G M_0 \oplus P,$$

where $\text{Ind}_H^G M_0 = M$ and $M_0 \cong J := \text{Ind}_{H_1}^H I_{H_1} \oplus \cdots \oplus \text{Ind}_{H_t}^H I_{H_t}$. We get

$$\text{Res}_H I_G = M_0 \oplus R \oplus P_1, \quad \text{Res}_H I_G = I_H \oplus P_2,$$

where $P_1 = \text{Res}_H P$ and P_2 are free Λ_H -modules, see (1.8)(i), and $M_0 \oplus R = \text{Res}_H \text{Ind}_H^G M$, R a Λ_H -module, see (1.8)(ii). Since $M_0 \cong J$ is finitely generated and has no free Λ_H -summand, it follows that $M_0 | I_H$ (use (1.2)).

Claim 1 : $\text{Ind}_{H_1}^H I_{H_1} \oplus \cdots \oplus \text{Ind}_{H_t}^H I_{H_t} = I_H$.

Proof : The canonical map

$$J = \text{Ind}_{H_1}^H I_{H_1} \oplus \cdots \oplus \text{Ind}_{H_t}^H I_{H_t} \xrightarrow{\text{can}} I_H$$

of Λ_H -modules induced by the inclusions is surjective as the subgroups H_i generate H . By the consideration above we have for the finitely generated Λ_H -module I_H a decomposition $I_H \cong M_0 \oplus T \cong J \oplus T$, T some Λ_H -module. Thus we get a surjection $J \xrightarrow{\text{can}} I_H \twoheadrightarrow J$. Since a surjective endomorphism of finitely generated Λ_H -modules is an isomorphism, it follows that $J \xrightarrow{\text{can}} I_H$ is bijective. This proves claim 1.

Thus we get an isomorphism $I_G \cong \text{Ind}_H^G I_H \oplus (\Lambda_G)^r$; but we have to show that the canonical inclusion $\text{Ind}_H^G I_H \hookrightarrow I_G$ splits.

Claim 2 : $I_G = \text{Ind}_H^G I_H \oplus P'$, P' a free Λ_G -module of rank r .

Proof: Again we consider $\text{Res}_H I_G = I_H \oplus P_2 = M_0 \oplus R \oplus P_1$. Since $I_H \oplus P_2$ is a direct sum of indecomposable, finitely generated Λ_H -modules (thus their endomorphism rings are local) and $M_0 \cong I_H$ is finitely generated, we have by [1] (12.2), (12.6):

$$I_H \oplus R \oplus P_1 = M_0 \oplus R \oplus P_1.$$

Thus the map $I_H \hookrightarrow M_0 \oplus (R \oplus P_1) \xrightarrow{pr} M_0$ is an isomorphism, and so

$$\text{Ind}_H^G I_H \hookrightarrow M \oplus \text{Ind}_H^G (R \oplus P_1) \xrightarrow{pr} M$$

is an isomorphism of Λ_G -modules. Let $\{(h_i - 1) + I_H^2, i = 1, \dots, d(H)\}$, $h_i \in H$, be a basis of $I_H/I_H^2 \cong H/H^*$, and let $h_i - 1 = m_i + p_i$, $m_i \in M_0$, $p_i \in (R \oplus P_1)$. Then $\{m_i + I_H M_0, i = 1, \dots, d(H)\}$ is a basis of $(M_0)_H$, and $\{m_i + I_G M, i = 1, \dots, d(H)\}$ is a basis of M_G . We have

$$I_G^2 = I_G M \oplus I_G P \subseteq I_G M \oplus \text{Ind}_H^G (R \oplus P_1).$$

It follows that $\{(h_i - 1) + I_G^2, i = 1, \dots, d(H)\} \subseteq I_G/I_G^2$ is a basis of $(I_H + I_G^2)/I_G^2 \subseteq I_G/I_G^2$. Indeed, this set is linearly independent: let $0 = \sum_{i=1}^{d(H)} a_i (h_i - 1) + I_G^2$, $a_i \in \mathbb{F}_p$. Then

$$\sum_{i=1}^{d(H)} a_i (h_i - 1) = \sum_{i=1}^{d(H)} a_i m_i + \sum_{i=1}^{d(H)} a_i p_i \in I_G M \oplus \text{Ind}_H^G (R \oplus P_1),$$

thus $0 = \sum_{i=1}^{d(H)} a_i m_i + I_G M$, and so $a_i = 0$ for all i .

This shows that the canonical map

$$H/H^* \xrightarrow[\sim]{can} HG^*/G^*$$

is an isomorphism and it follows that $d(G) - \dim_{\mathbb{F}_p} HG^*/G^* = r$.

Let $\{g_1 G^*, \dots, g_r G^*\} \subseteq G/G^*$ such that $\{h_1 G^*, \dots, h_{d(H)} G^*, g_1 G^*, \dots, g_r G^*\}$ is a basis of G/G^* and let

$$\text{Ind}_H^G I_H \oplus P \xrightarrow{(can, \varphi)} I_G$$

be the surjective map defined by the canonical inclusion on the first summand and the homomorphism φ which maps a basis of P onto $\{g_1 - 1, \dots, g_r - 1\} \subseteq I_G$. Since $I_G = M \oplus P \cong \text{Ind}_H^G I_H \oplus P$ and since a surjective endomorphism of finitely generated Λ_G -modules is an isomorphism, it follows that the map above is an isomorphism. This proves claim 2.

Using (1.10), we obtain

$$I_G = \text{Ind}_H^G I_H \oplus \text{Ind}_F^G I_F,$$

where F is a free pro- p subgroup of G of rank r . Let $H_i \cong \mathbb{Z}_p$, $i = t + 1, \dots, s$, such that $F = H_{t+1} * \dots * H_s$. It follows that the canonical map

$$\text{Ind}_{H_1}^G I_{H_1} \oplus \dots \oplus \text{Ind}_{H_s}^G I_{H_s} \xrightarrow{\sim} I_G$$

induced by the inclusions is an isomorphism of Λ_G -modules. Using (1.6), we see that the map

$$H_1 * \dots * H_s \xrightarrow{\sim} G$$

induced by the inclusion is bijective.

It remains to show that the groups H_i are freely indecomposable. Suppose one of the H_i decomposes freely, then by (1.6) the augmentation ideal I_G decomposes in more than s non-trivial summands, which is impossible by the Krull-Schmidt-Azumaya theorem (1.1)(ii). This finishes the proof of the theorem. \square

2 Free pro- p products

A finitely generated (abstract) group admits a decomposition into a free product of freely indecomposable groups, called its *Grushko decomposition*. The following theorem is a pro- p analog of this result.

Theorem 2.1

- (i) *Every finitely generated pro- p group G is the free pro- p product of finitely many freely indecomposable closed subgroups G_i , $i = 1, \dots, s$, i.e.*

$$G = \bigstar_{i=1}^s G_i.$$

- (ii) *Let $G = \bigstar_{i \in I} G_i$ be the free pro- p product of pro- p groups G_i , $i \in I$, and let H be a finitely generated closed subgroup of G which is freely indecomposable. Then $H \cong \mathbb{Z}_p$ or there exist $j \in I$ and $\sigma \in G$ such that $H \subseteq (G_j)^\sigma$.*

- (iii) *Let*

$$G = \bigstar_{i=1}^n G_i * F_t = \bigstar_{j=1}^m H_j * F_u$$

be two decompositions of the finitely generated pro- p group G , where the closed subgroups H_j and G_i are freely indecomposable and not isomorphic to \mathbb{Z}_p and F_t and F_u are free pro- p groups of rank t and u , respectively. Then $n = m$, $t = u$ and there are elements $\sigma_1, \dots, \sigma_n$ in G such that (after possibly re-ordering) $G_i = (H_i)^{\sigma_i}$ for $i = 1, \dots, n$.

*In particular, the number $s(G) = n + t$ of freely indecomposable factors of G is an invariant of G . We call a decomposition as above a **Grushko decomposition**.*

Proof: (i) In contrast to the theory of abstract groups, for pro- p groups it is easy to see that the generator rank $d(H_1 * H_2)$ of a free pro- p product $H_1 * H_2$ is $d(H_1) + d(H_2)$, see [12](3.9.1),(4.1.4), and so assertion (i) is obvious.

(ii) follows from the pro- p analog of Kurosh' subgroup theorem for *finitely generated closed* subgroups of free pro- p products, see [7] Theorem (9.7) or [8] Theorem (4.4) or [11] Proposition (5.2).

Now we obtain (iii) easily: Using (ii), we get

$$G_i \subseteq (H_j)^{\sigma_j} \subseteq (G_k)^{\tau_k},$$

for $\sigma_j, \tau_k \in G$. It follows that $G_i = (G_k)^{\tau_k}$, hence $G_i = (H_j)^{\sigma_j}$, and so $n = m$. Dividing out the normal closure of the subgroup generated by the H_i 's, we get an isomorphism $F_t \cong F_u$, hence $t = u$. \square

Lemma 2.2 *Let*

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\psi} G/H \longrightarrow 1$$

be an exact sequence of pro- p groups, where $G/H \cong \mathbb{F}_p$.

- (i) *Let G be free of rank r with basis $\{\tau_1, \dots, \tau_r\}$ such that ψ maps τ_1 to a generator of G/H and $\psi(\tau_i) = 1$ for $i \geq 2$. Then H is free of rank $p(r-1)+1$ with basis*

$$\{\tau_1^p, (\tau_i)^{(\tau_1)^k}, k = 0, \dots, p-1, i = 2, \dots, r\},$$

and there is a $\mathbb{Z}_p[G/H]$ -isomorphism

$$H^{ab} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p[G/H]^{r-1}.$$

- (ii) *Let*

$$G = \bigstar_{i=1}^n G_i * L,$$

$n \geq 1$, and each factor G_i of the free pro- p product is mapped surjectively onto G/H and L to 1. Then there is a decomposition of H as free pro- p product

$$H = \tilde{H} * F_d, \quad \tilde{H} = \bigstar_{i=1}^n H_i * \bigstar_{j=0}^{p-1} L^{(\tau_1)^j},$$

where $H_i = H \cap G_i$, $\tau_i \in G_i \setminus H_i$, $i = 1, \dots, n$, and F_d is free of rank $(p-1)(n-1)$ with basis $\{(\tau_i \tau_1^{-1})^{(\tau_1)^k}, i \geq 2, k = 1, \dots, p-1\}$.

Furthermore, if M is the normal subgroup of H generated by \tilde{H} and $\bar{F}_d = F_d M / M$, then there is a $\mathbb{Z}_p[G/H]$ -isomorphism

$$(\bar{F}_d)^{ab} \cong (I_{G/H})^{n-1}.$$

Proof: (ii) The pro- p analog of Kurosh' subgroup theorem for open subgroups of free pro- p products, see [12] (4.2.1), implies that H has the asserted structure. In order to find a basis of F_d , we first consider the case $G_i = \langle \tau_i \rangle \cong \mathbb{Z}_p$, $i = 1, \dots, n$, and $L = 1$. Schreier's subgroup theorem of free groups, see [14] (3.6.2)(a) for the profinite case, implies that

$$u_1 = \tau_1^p, \quad u_{i,k} = (\tau_i \tau_1^{-1})^{(\tau_1)^k}, \quad u_{i,p-1} = (\tau_1)^{p-1} \tau_i = (\tau_i \tau_1^{-1})^{(\tau_1)^{p-1}} \tau_1^{-p},$$

$k = 0, \dots, p-2$, $i = 2, \dots, n$, is a basis of H . Changing the basis, we get

$$u_1 = \tau_1^p, \quad u_{i,k} = (\tau_i \tau_1^{-1})^{(\tau_1)^k}, \quad u_i := u_{i,0} \cdots u_{i,p-2} u_{i,p-1} = \tau_i^p,$$

hence $H_i = \langle \tau_i^p \rangle$, $i \geq 1$, and $\{(\tau_i \tau_1^{-1})^{(\tau_1)^k}, i \geq 2, k = 1, \dots, p-1\}$ is a basis of F_d .

Dividing out the normal subgroup generated by the subgroups H_i (which is also normal in G), we are in the case where $G_1 = \cdots = G_n \cong \mathbb{F}_p$ and $L = 1$. In general, we also obtain this case by dividing out the normal subgroup M (which is also normal in G), and get so the desired result.

Considering the explicitly given basis of F_d , we get the asserted structure of the $\mathbb{Z}_p[G/H]$ -module $(\bar{F}_d)^{ab}$, or one can see this as follows. First observe that $\bar{F}_d \cong F_d$ and so $(\bar{F}_d)^{ab}$ is \mathbb{Z}_p -free. We have $G/M = \bigstar_{i=1}^n G_i/H_i$ and therefore the exact sequence

$$1 \longrightarrow \bar{F}_d \longrightarrow \bigstar_{i=1}^n G_i/H_i \longrightarrow G/H \longrightarrow 1,$$

inducing the exact sequence

$$0 \longrightarrow (\bar{F}_d^{ab})_{G/H} \longrightarrow \bigoplus_{i=1}^n G_i/H_i \longrightarrow G/H \longrightarrow 0.$$

It follows that $(\bar{F}_d^{ab})_{G/H} \cong \mathbb{F}_p^{n-1}$. Since

$$(\bar{F}_d)^{ab} \cong \mathbb{Z}_p[G]^a \oplus (I_G)^b \oplus \mathbb{Z}_p^c,$$

where

$$(p-1)(n-1) = pa + (p-1)b + c,$$

see [6] §30C, we obtain $(\bar{F}_d)^{ab} \cong (I_G)^b$ and $b = n-1$.

Finally we see that (i) is a special case of (ii): take $n = 1$, $G_1 = \langle \tau_1 \rangle$ and $L = \langle \tau_2, \dots, \tau_r \rangle$. \square

The following theorem is in some sense the converse of lemma (2.2)(ii), i.e. the converse of a special case of Kurosh' subgroup theorem; we will use the notion $N_G(L)$ for the normalizer of a closed subgroup L in G .

Theorem 2.3 *Let G be a finitely generated torsion-free pro- p group and let H be an open normal subgroup such that $G/H \cong \mathbb{F}_p$. Then H and G have a Grushko decomposition as a free pro- p product of closed subgroups of the following form:*

$$G = \bigstar_{i=1}^n G_i * \bigstar_{j=1}^m L_j * F_\delta,$$

$$H = \bigstar_{i=1}^n H_i * \bigstar_{j=1}^m \bigstar_{\sigma \in G/H} (L_j)^\sigma * F_d,$$

$n \geq 0$, $m \geq 0$, where $G_i = N_G(H_i)$ and

H_i is freely indecomposable not isomorphic to \mathbb{Z}_p and $G_i/H_i \cong \mathbb{F}_p$,
 L_j is freely indecomposable and $N_G(L_j) = L_j$,
 F_d and F_δ are free of rank $d = (p-1)(n-1)$ and $\delta = 0$, respectively,
if $n \geq 1$, and $d = \delta = 1$ otherwise.

The following corollary would be an immediate consequence of the main theorem of the introduction, but we have to use it in order to prove this theorem.

Corollary 2.4 *Let G be a finitely generated torsion-free pro- p group and let H be an open subgroup. Then H is freely decomposable if and only if G is freely decomposable.*

Remarks: 1. The assumption that G has to be torsion-free is necessary as the following example shows: Let $G = F_2 \times \mathbb{Z}/p\mathbb{Z}$, where F_2 is a free pro- p group of rank 2, and $H = F_2$. Then H is freely decomposable, but G not.

2. Also the opposite case is interesting: The group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2 \rtimes \mathbb{Z}/2\mathbb{Z}$ is decomposable and has an open subgroup which is freely indecomposable. Kurosh' subgroup theorem shows that this is the only pro- p group with this property.

Proof of (2.4): The pro- p analog of Kurosh' subgroup theorem for open subgroups of free pro- p products, see [12] (4.2.1), implies the if-part. In order to show the converse we may assume that H is normal in G of index p . Now the result follows from (2.3). \square

Proof of (2.3): Let

$$H = \bigstar_{i=1}^n H_i * \bigstar_{\lambda=1}^\mu K_\lambda * F_u,$$

be a decomposition of H , where H_i and K_λ are freely indecomposable closed subgroups not isomorphic to \mathbb{Z}_p , $N_G(H_i) \neq H_i$, $i = 1, \dots, n$, and $N_G(K_\lambda) = K_\lambda$, $\lambda = 1, \dots, \mu$, and F_u is free of rank u . Since H is finitely generated and $d(H) = \sum_i d(H_i) + \sum_\lambda d(K_\lambda) + u$, the groups H_i and K_λ are finitely generated, too. Let

$G_i = N_G(H_i)$. As $N_H(H_i) = H_i$, see [14] (9.1.12), we have $H \cap G_i = H_i$, hence a commutative and exact diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & \mathbb{F}_p \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & H_i & \longrightarrow & G_i & \longrightarrow & \mathbb{F}_p \longrightarrow 0. \end{array}$$

It follows from Kurosh' subgroup theorem that G_i is freely indecomposable. Furthermore, since $K_\lambda \neq (K_\lambda)^\sigma$ for a representative $\sigma \in G$ of $\bar{\sigma} \in G/H$, $\bar{\sigma} \neq 1$, it follows from (2.1)(ii) that $(K_\lambda)^\sigma$ is a H -conjugate of $K_{\lambda'}$ for some $\lambda' \neq \lambda$. Therefore, we can assume that the above decomposition is of the form

$$H = H' * F_u, \text{ where } H' = \bigstar_{i=1}^n H_i * \bigstar_{j=1}^{m'} \bigstar_{\sigma \in G \setminus H} (K_j)^\sigma.$$

Observe that the normal closure M of H' in H is also normal in G , and so we obtain an exact sequence $1 \rightarrow H/M \rightarrow G/M \rightarrow \mathbb{F}_p \rightarrow 1$. Since $H/M \cong F_u$, the group $(H/M)^{ab}$ is \mathbb{Z}_p -free and there is a $\mathbb{Z}_p[G/H]$ -isomorphism

$$(H/M)^{ab} \cong \mathbb{Z}_p[G/H]^{u_1} \oplus (I_{G/H})^{u_2} \oplus (\mathbb{Z}_p)^{u_3},$$

where $u = pu_1 + (p-1)u_2 + u_3$, see [6] §30C. It follows that H is of the form

$$H = \tilde{H} * F_{(p-1)u_2 + u_3},$$

$$\tilde{H} = \bigstar_{i=1}^n H_i * \bigstar_{\sigma \in G \setminus H} \left(\bigstar_{j=1}^{m'} K_j * F_{u_1} \right)^\sigma = \bigstar_{i=1}^n H_i * \bigstar_{\sigma \in G \setminus H} \bigstar_{j=1}^m (L_j)^\sigma;$$

here L_j is a freely indecomposable factor of the form K_j or \mathbb{Z}_p and $m = m' + u_1$. In the following we put $u = (p-1)u_2 + u_3$ and M now denotes the normal closure of \tilde{H} in H . If we consider a subgroup U of G modulo M , we denote it by \bar{U} . We have a $\mathbb{Z}_p[G/H]$ -isomorphism

$$(\bar{F}_u)^{ab} \cong (I_{G/H})^{u_2} \oplus (\mathbb{Z}_p)^{u_3}.$$

Let

$$G_0 = \begin{cases} \bigstar_{i=1}^n G_i * \bigstar_{j=1}^m L_j, & \text{if } n > 0, \\ \bigstar_{j=1}^m L_j * \Gamma, & \text{if } n = 0, \end{cases}$$

where $\Gamma = \mathbb{Z}_p$, and let

$$\varphi : G_0 \longrightarrow G$$

be the homomorphism which is the inclusion on each factor G_i or L_j and in the second case a generator of Γ is mapped to a pre-image in G of a generator of G/H . We obtain a commutative and exact diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H \longrightarrow 1 \\ & & \uparrow & & \uparrow \varphi & & \parallel \\ 1 & \longrightarrow & H_0 & \longrightarrow & G_0 & \longrightarrow & G/H \longrightarrow 1, \end{array}$$

where

$$H_0 = \tilde{H} * F_d,$$

with $d = (p-1)(n-1)$, if $n \geq 1$ and $d = 1$ otherwise. In the first case it follows that there is a $\mathbb{Z}_p[G/H]$ -isomorphism

$$(\bar{F}_d)^{ab} \cong (I_{G/H})^{n-1},$$

see (2.2)(ii) and observe that $\bar{F}_d = F_d N / N$ where N denotes the normal closure of \tilde{H} in H_0 .

Part 1: Let $\tau \in H$. Then the subgroup $U = \langle \tau, F_u \rangle \subseteq H$ is free.

Proof: Let $U = V * F_t$, $V = \ast_{k \in I} V_k$, be a free decomposition of U , where F_t is free of rank t and the groups V_k are freely indecomposable not isomorphic to \mathbb{Z}_p . Since G is torsion-free and so U is, we have $d(V_k) > 1$ if V_k is not trivial. By the Kurosh subgroup theorem for finitely generated closed subgroups it follows that $V \subseteq M$ (recall that M is the normal closure of \tilde{H} in H), hence $UM/M = F_t M/M$. Since $UM/M \cong \bar{F}_u$, it follows that $t \geq u$, and since $d(U) \leq u + 1$, we obtain that $U = F_t$, where $u \leq t \leq u + 1$.

Part 2: φ is surjective.

Proof: We consider the commutative and exact diagram

$$\begin{array}{ccccccc} & & D & \twoheadrightarrow & C & & \\ & & \uparrow & & \uparrow & & \\ (\tilde{H} * F_u)^{ab} & \longrightarrow & G^{ab} & \longrightarrow & G/H & \longrightarrow & 0 \\ & & \uparrow \bar{\varphi} & & \parallel & & \\ (\tilde{H} * F_d)^{ab} & \longrightarrow & G_0^{ab} & \longrightarrow & G/H & \longrightarrow & 0, \end{array}$$

where the map $\bar{\varphi}$ is induced by φ . The cokernel D is an image of \bar{F}_u^{ab} and $C = D_G$. Hence we have an exact sequence

$$(\bar{F}_d / \bar{F}_d^*)_G \longrightarrow (\bar{F}_u / \bar{F}_u^*)_G \longrightarrow C / C^* \longrightarrow 0.$$

If φ is not surjective, then we need additional generators: Let

$$\varphi': G_0 * F_s \twoheadrightarrow G$$

be surjective, where F_s is free of rank s , and a basis of F_s is mapped onto generators x_1, \dots, x_s of G which are pre-images of a basis of C/C^* and contained in $F_u \setminus \varphi(F_d) \subseteq H$. Let $\sigma \in G_0$ be a pre-image of a generator of G/H under the map $G_0 \xrightarrow{\varphi} G \twoheadrightarrow G/H$, which is contained in $G_1 \subseteq G_0$ if $n \geq 1$ and in Γ otherwise. We denote $\varphi'(\sigma)$ also by σ . Consider the following subgroup

$$F_0 = \langle \sigma \rangle * F_s$$

of $G_0 * F_s$ and the homomorphism $\psi: F_0 \hookrightarrow G_0 * F_s \twoheadrightarrow G \twoheadrightarrow G/H$. Using (2.2)(i), we have the exact sequence

$$1 \longrightarrow E_0 \longrightarrow F_0 \xrightarrow{\psi} G/H \longrightarrow 1,$$

where

$$E_0 = \langle \sigma^p \rangle * E'_0, \quad E'_0 = \bigstar_{i=0}^{p-1} F_s^{\sigma^i},$$

and a $\mathbb{Z}_p[G/H]$ -isomorphism

$$(\bar{E}'_0)^{ab} = \mathbb{Z}_p[G/H]^s.$$

Furthermore, let $F = \varphi'(F_0)$ and $E = \varphi'(E_0) = F \cap H$. Then

$$E = \langle \sigma^p, E' \rangle$$

where $E' \subseteq F_u$ is the pre-image of $\varphi'(E'_0)M/M \subseteq \bar{F}_u$. We have the commutative diagrams

$$\begin{array}{ccccccc} 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \parallel \\ 1 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & G/H \longrightarrow 1 \\ & & \uparrow & & \uparrow \varphi' & & \parallel \\ 1 & \longrightarrow & E_0 & \longrightarrow & F_0 & \longrightarrow & G/H \longrightarrow 1 \end{array}$$

and

$$\begin{array}{ccccc} E_0 = \langle \sigma^p \rangle * E'_0 & \twoheadrightarrow & E = \langle \sigma^p, E' \rangle & \twoheadrightarrow & EM/M = \bar{E} \\ & & \downarrow & & \downarrow \\ & & H = \tilde{H} * F_u & \twoheadrightarrow & H/M = \bar{F}_u. \end{array}$$

By part 1 the group E is free. Since F as subgroup of G is torsion-free, we obtain from a theorem of Serre [15] that F is free.

The generator rank of F is $d(F) = 1 + s$, as $F/F^* \cong \langle \bar{\sigma} \rangle \oplus C/C^*$, where $\bar{\sigma} = \sigma F^*$. It follows that $d(E) = ps + 1$, hence $E'_0 \simeq E'$ and so $(\bar{E}'_0)^{ab} \cong \mathbb{Z}_p[G/H]^s$. By definition of F_s and since C/C^* is a direct summand of \bar{F}_u/\bar{F}_u^* , it follows that

$(\bar{E}')^{ab} \subseteq (\bar{F}_u)^{ab}$ is a direct \mathbb{Z}_p -summand. Thus we have an exact sequence of $\mathbb{Z}_p[G/H]$ -modules

$$0 \longrightarrow (\bar{E}')^{ab} \longrightarrow (\bar{F}_u)^{ab} \longrightarrow R \longrightarrow 0,$$

where $R = (\bar{F}_u)^{ab}/(\bar{E}')^{ab}$, which splits as a sequence of free \mathbb{Z}_p -modules, i.e. $\text{Ext}_{\mathbb{Z}_p}^1(R, (\bar{E}')^{ab}) = 0$. Therefore

$$\text{Ext}_{\mathbb{Z}_p[G/H]}^1(R, (\bar{E}')^{ab}) = H^1(G/H, \text{Hom}_{\mathbb{Z}_p}(R, (\bar{E}')^{ab})) = 0,$$

as $\text{Hom}_{\mathbb{Z}_p}(R, (\bar{E}')^{ab})$ is a cohomological trivial $\mathbb{Z}_p[G/H]$ -module. We obtain a $\mathbb{Z}_p[G/H]$ -isomorphism

$$R \oplus \mathbb{Z}_p[G/H]^s \cong (\bar{F}_u)^{ab} \cong (I_{G/H})^{u_2} \oplus (\mathbb{Z}_p)^{u_3},$$

hence $s = 0$. This proves part 2.

Part 3: Let $n \geq 1$, then $d(G) = d(G_0)$.

Proof: We consider the surjection

$$G_0/G_0^* = \bigoplus_{i=1}^n G_i/G_i^* \oplus \bigoplus_{j=1}^m L_j/L_j^* \twoheadrightarrow G/G^*.$$

Suppose that $d(G) < d(G_0)$. Then one summand in G_0/G_0^* can be replaced by a 1-codimensional subspace and the corresponding map remains to be surjective. The only possible replacement is G_i/G_i^* by $H_i G_i^*/G_i^*$, as otherwise the corresponding map $U \rightarrow G$, where $U \subseteq G_0$ is a pre-image of the 1-codimensional subspace of G_0/G_0^* obtained by this replacement, would not induce a surjection onto H . Thus there would be a surjection (after re-ordering)

$$\varphi' : G'_0 = H_1 * \bigstar_{i=2}^n G_i * \bigstar_{j=1}^m L_j \twoheadrightarrow G,$$

and n has to be bigger or equal to 2. Let $\sigma_j \in G_j \setminus H_j$, $j = 1, \dots, n$. Using the Kurosh subgroup theorem for open subgroups, we obtain the commutative and exact diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \bigstar_{i=1}^n H_i * \bigstar_{k=0}^{p-1} \bigstar_{j=1}^m (L_j)^{(\sigma_2)^k} * F_u & \longrightarrow & G & \longrightarrow & \mathbb{F}_p \longrightarrow 1 \\ & & \uparrow & & \uparrow \varphi' & & \parallel \\ 1 & \longrightarrow & \bigstar_{k=0}^{p-1} (H_1)^{(\sigma_2)^k} * \bigstar_{i=2}^n H_i * \bigstar_{k=0}^{p-1} \bigstar_{j=1}^m (L_j)^{(\sigma_2)^k} * F_{d'} & \longrightarrow & G'_0 & \longrightarrow & \mathbb{F}_p \longrightarrow 1, \end{array}$$

where $u = (p-1)u_2 + u_3$, $d' = (p-1)(n-2)$ and $F_{d'}$ is generated by the elements $(\sigma_j \sigma_2^{-1})^{(\sigma_2)^k}$, $j = 3, \dots, n$, $k = 1, \dots, p-1$, see (2.2)(ii).

Let N' be the normal closure of $\ast_{i=2}^n G_i \ast \ast_{j=1}^m L_j$ in G'_0 and let M' be its image in G . From the commutative diagram above it follows that the kernel of φ' is contained in the normal closure of the subgroup generated by $(H_1)^{(\sigma_2)^{k-1}}$, $k = 1, \dots, p-1$, and $F_{d'}$, in particular in N' . Therefore we get a commutative and exact diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & M' & \longrightarrow & G & \longrightarrow & H_1 \longrightarrow 1 \\ & & \uparrow & & \uparrow \varphi' & & \parallel \\ 1 & \longrightarrow & N' & \longrightarrow & G'_0 & \xrightarrow{\sim} & H_1 \longrightarrow 1. \end{array}$$

It follows that G is the semi-direct product of H_1 and M' . We can assume that the element $\sigma_1 \in G_1 \setminus H_1$ is chosen such that its pre-image in G'_0 lies inside N' . Then $(\sigma_1)^p$ is contained in $H_1 \cap M'$, hence $(\sigma_1)^p = 1$. Thus G contains a torsion element, which is a contradiction. This finishes the proof of part 3.

Part 4: Let $n \geq 1$, then φ is an isomorphism.

Proof: We have a commutative and exact diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H \longrightarrow 1 \\ & & \uparrow & & \uparrow \varphi & & \parallel \\ 1 & \longrightarrow & H_0 & \longrightarrow & G_0 & \longrightarrow & G/H \longrightarrow 1, \\ & & \uparrow & & \uparrow & & \\ & & K & \xlongequal{\quad} & K & & \end{array}$$

where $K = \text{Ker } \varphi$, $H_0 = \tilde{H} \ast F_d$, $d = (p-1)(n-1)$. Recall that N is the normal closure of \tilde{H} in H_0 (which is also normal in G_0) and M is the normal closure of \tilde{H} in H (which is also normal in G). If we consider a subgroup U of G_0 modulo N , we denote it by \bar{U} . We have a $\mathbb{Z}_p[G/H]$ -isomorphism

$$(\bar{F}_d)^{ab} \cong (I_{G/H})^{n-1}.$$

Furthermore, since the inflation map $H^2(H, \mathbb{Q}_p/\mathbb{Z}_p) \simeq H^2(H_0, \mathbb{Q}_p/\mathbb{Z}_p)$ is an isomorphism, the exact sequence $1 \rightarrow K \rightarrow H_0 \rightarrow H \rightarrow 1$ induces the exact sequence

$$0 \rightarrow (K^{ab})_H \rightarrow (H_0)^{ab} \rightarrow H^{ab} \rightarrow 0.$$

Since $(N^{ab})_{H_0} \xrightarrow{\bar{\varphi}} (M^{ab})_H$, the commutative and exact diagram

$$\begin{array}{ccccccc} & & & & (K^{ab})_H & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & (N^{ab})_{H_0} & \longrightarrow & (H_0)^{ab} & \longrightarrow & (\bar{F}_d)^{ab} \longrightarrow 0 \\ & & \downarrow \sim & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (M^{ab})_H & \longrightarrow & H^{ab} & \longrightarrow & (\bar{F}_u)^{ab} \longrightarrow 0 \end{array}$$

yields the exact sequence

$$0 \rightarrow (K^{ab})_H \rightarrow (\bar{F}_d)^{ab} \rightarrow (\bar{F}_u)^{ab} \rightarrow 0,$$

i.e. the exact sequence

$$0 \rightarrow (K^{ab})_H \rightarrow (I_{G/H})^{n-1} \rightarrow (I_{G/H})^{u_2} \oplus \mathbb{Z}_p^{u_3} \rightarrow 0.$$

It follows that u_3 has to be zero, since $((I_{G/H})^{n-1})_{G/H} = \mathbb{F}_p^{n-1}$ surjects onto $\mathbb{Z}_p^{u_3}$, and $(K^{ab})_H \cong (I_{G/H})^s$, $s = n - 1 - u_2$, since it has no factor isomorphic to \mathbb{Z}_p or $\mathbb{Z}_p[G/H]$ as contained in $(I_{G/H})^{n-1}$. The exact and commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & G/M \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & N & \longrightarrow & G_0 & \longrightarrow & G_0/N \longrightarrow 1 \end{array}$$

yields the exact and commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & MG^*/G^* & \longrightarrow & G/G^* & \longrightarrow & (G/M)/(G/M)^* \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & NG_0^*/G_0^* & \longrightarrow & G_0/G_0^* & \longrightarrow & (G_0/N)/(G_0/N)^* \longrightarrow 0. \end{array}$$

Using part 3, it follows that $d(G/M) = d(G_0/N) = n$. Now the exact sequence

$$1 \longrightarrow \bar{F}_{(p-1)u_2} \longrightarrow G/M \longrightarrow G/H \longrightarrow 1$$

induces the exact sequence

$$0 \longrightarrow (\bar{F}_{(p-1)u_2})_{G/H}^{ab} \longrightarrow (G/M)^{ab} \longrightarrow G/H \longrightarrow 0,$$

i.e. the exact sequence

$$0 \longrightarrow \mathbb{F}_p^{u_2} \longrightarrow (G/M)^{ab} \longrightarrow \mathbb{F}_p \longrightarrow 0.$$

Since $(G_0/N)^{ab}$ is elementary abelian, the same is true for its homomorphic image $(G/M)^{ab}$. It follows that $d(G/M) = u_2 + 1$. Hence $u_2 = n - 1$, and so $(K^{ab})_{G/H} = 0$. It follows that $K = 1$, i.e. φ is bijective. This proves part 4.

Part 5: Let $n = 0$, then φ is an isomorphism.

Proof: Let $\sigma \in G$ be a pre-image of a generator of G/H and $L = \bigstar_{j=1}^m L_j$. Consider the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \bigstar_{i=0}^{p-1} L^{\sigma^i} * F_u & \longrightarrow & G & \longrightarrow & G/H \longrightarrow 1 \\ & & \uparrow & & \uparrow \varphi & & \parallel \\ 1 & \longrightarrow & \bigstar_{i=0}^{p-1} L^{\sigma^i} * \Gamma^p & \longrightarrow & L * \Gamma & \longrightarrow & G/H \longrightarrow 1. \end{array}$$

Since φ is surjective, and so $\Gamma^p \cong \mathbb{Z}_p \twoheadrightarrow (\bar{F}_u)^{ab}$ is a surjection of $\mathbb{Z}_p[G/H]$ -modules, we obtain $u_2 = 0$ and $u_3 \leq 1$. Suppose $u_3 = 0$. Then $H = \ast_{i=0}^{p-1} L^{\sigma^i}$ and the exact sequence

$$0 \longrightarrow H^{ab} \longrightarrow G/[H, H] \longrightarrow G/H \longrightarrow 0$$

splits, because of $H^2(G/H, H^{ab}) = 0$. It follows that $G/G^* \cong G/H \oplus (H/H^*)_{G/H}$, i.e. $d(G) = d(L) + 1$.

Let G_1 be a finitely generated torsion-free and freely indecomposable pro- p group not isomorphic to \mathbb{Z}_p having a surjection $G_1 \twoheadrightarrow \mathbb{Z}_p$ (e.g. $G_1 = \mathbb{Z}_p \oplus \mathbb{Z}_p$), and let $G' = G_1 * G$ and $\psi: G' \twoheadrightarrow G$ be the homomorphism which is given by the identity on G and the homomorphism $\pi: G_1 \twoheadrightarrow \mathbb{Z}_p \rightarrow G$ mapping a generator of \mathbb{Z}_p to σ . Let $H_1 \subseteq G_1$ be the kernel of the surjection $G_1 \xrightarrow{\pi} G \twoheadrightarrow G/H$. Using the Kurosh subgroup theorem, we get an exact and commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & G/H \longrightarrow 1 \\ & & \uparrow & & \uparrow \psi & & \parallel \\ 1 & \longrightarrow & H_1 * H * F_{p-1} & \longrightarrow & G_1 * G & \longrightarrow & G/H \longrightarrow 1, \end{array}$$

Considering the group G' instead of G , we are in the situation $n \geq 1$ because of $G_1 = N_{G'}(H_1) \neq H_1$, and we can use the result obtained in that case. Hence

$$G' = G_1 * G \cong G_1 * L$$

which is obviously a contradiction as $d(G') = d(G) + d(G_1) = d(L) + 1 + d(G_1)$. Thus $u_3 = 1$ and it follows that the surjection $H_0 \twoheadrightarrow H$ is bijective, i.e. φ is an isomorphism. This finishes the proof of the theorem. \square

Proof of Theorem 2: The equivalence (i) \Leftrightarrow (ii) is theorem (1.14).

In order to prove (iii) \Rightarrow (iv) let H be an open subgroup of G such that all open subgroups $H' \subseteq H$ are decomposable, and so by (1.6) the augmentation ideals $I_{H'}$ are decomposable. Using (1.4) (i) and (iv), it follows that $h_1(G) = h_1(H') = \infty$.

Assuming (iv), it follows that there is an open subgroup H of G such that I_H has a direct summand isomorphic to Λ_H , see (1.4)(ii). Suppose that $I_H \cong \Lambda_H$, then by (1.4)(iii) $H \cong \mathbb{Z}_p$ and $h^1(G) = h^1(H) = 1$, a contradiction. It follows that $I_H \cong M \oplus \Lambda_H$, where M is a non-trivial left Λ_H -module. From theorem (1.14) and (1.10) it follows that $H = H_0 * F_1$ is freely decomposable, H_0 a non-trivial closed subgroup of H and $F_1 \cong \mathbb{Z}_p$. Using Kurosh' subgroup theorem we see that all open subgroups of H are freely decomposable.

If G is torsion free, then by (2.4) we have (i) \Leftrightarrow (iii). \square

Proof of Corollary 1: From theorem 2 and (1.3)(ii) it follows that G is freely indecomposable if and only if H is. Let

$$G = \bigstar_{i=1}^{s(G)} G_i$$

be a decomposition of G into freely indecomposable closed subgroups G_i . Recall the pro- p analog of Kurosh' subgroup theorem for open subgroups of free pro- p products, see [12] (4.2.1): There exist systems S_i of representatives s_i of the double coset decomposition $G = \bigcup_{s_i \in S_i} H s_i G_i$ for all i and a free pro- p group F_r of the finite rank

$$r = \sum_{i=1}^{s(G)} [(G : H) - \#S_i] - (G : H) + 1,$$

such that the natural inclusions induce a free product decomposition

$$H = \bigstar_{i=1}^{s(G)} \bigstar_{s_i \in S_i} (G_i^{s_i} \cap H) * F_r.$$

Since $G_i^{s_i} \cap H$ is an open subgroup of $G_i^{s_i}$, it is freely indecomposable by (2.4), hence

$$s(H) = \sum_{i=1}^{s(G)} \#S_i + \sum_{i=1}^{s(G)} [(G : H) - \#S_i] - (G : H) + 1 = (G : H)(s(G) - 1) + 1.$$

This proves part (ii) of the corollary. The proof of part (i) follows from proposition (2.5) below. \square

Proof of Corollary 2: Recall that G is a duality group of dimension 2 if and only if $D_1(G, \mathbb{F}_p) = 0$, i.e. $h^1(G) = 0$, see [12] (3.4.6). According to theorem 2 this is equivalent to the indecomposability of G resp. I_G .

Now let G be a torsion-free 2-generator group. If it would be freely decomposable, then it would be free. Thus it is freely indecomposable if $\text{cd}_p G = 2$, and so it is a duality group.

Proposition 2.5 *Let G be a finitely generated torsion-free pro- p group. Let*

$$G = \bigstar_{i=1}^t H_i * F_r.$$

be a decomposition of G as free pro- p product, where H_i , $i = 1, \dots, t$, are freely indecomposable closed subgroups, which are not free, and F_r is a free group of rank r , i.e. $s(G) = t + r$. Let

$$I_G = \bigoplus_{i=1}^{t'} M_i \oplus P_{r'}$$

be a decomposition of I_G into indecomposable left Λ_G -modules M_i , $i = 1, \dots, t'$, which are not isomorphic to Λ_G , and a free Λ_G -module $P_{r'}$ of rank r' . Then we have the following assertions.

(i) Let G be free. Then $t = t' = 0$, $r = r' = d(G)$,

$$s(G) = f(G),$$

the Λ_G -module $\text{Hom}_{\Lambda_G}(I_G, \Lambda_G)$ is free of rank $d(G) = f(G)$ and there is an exact sequence $0 \longrightarrow \Lambda_G \longrightarrow \Lambda_G^{f(G)} \longrightarrow H^1(G, \Lambda_G) \longrightarrow 0$.

(ii) If G is not free, then $t = t'$, $r = r'$,

$$s(G) = f(G) + 1$$

and the Λ_G -modules $\text{Hom}_{\Lambda_G}(I_G, \Lambda_G)$ and $H^1(G, \Lambda_G)$ are free of rank $f(G) + 1$ and $f(G)$, respectively. Furthermore, up to a re-ordering there are isomorphisms $M_i \cong J_{H_i}$, $i = 1, \dots, t$, of left Λ_G -modules.

Proof: Since $I_{F_r} \cong (\Lambda_{F_r})^r$, [12] (5.6.3), (5.6.4), we have $J_{F_r} \cong (\Lambda_G)^r$. Using (1.6), we obtain

$$I_G = \bigoplus_{i=1}^t J_{H_i} \oplus (\Lambda_G)^r.$$

Since H_i is not free, we have

$$\text{Ext}_G^1(J_{H_i}, \mathbb{F}_p) \cong \text{Ext}_{H_i}^1(I_{H_i}, \mathbb{F}_p) \cong H^2(H_i, \mathbb{F}_p) \neq 0,$$

hence J_{H_i} is not a free Λ_G -module.

If G is free of rank $d(G)$, then we have an isomorphism $I_G \cong \Lambda_G^{d(G)}$, [12] (5.6.3), (5.6.4). Furthermore, since G is a duality group with dualizing module $\varinjlim_n D_1(G, \mathbb{Z}/p^n\mathbb{Z})$ and since $H^1(G, \Lambda_G) = D_1(G, \mathbb{Z}/p\mathbb{Z})^\vee$, we obtain

$$H^1(G, \Lambda_G)_G = H^0(G, D_1(G, \mathbb{Z}/p\mathbb{Z}))^\vee \cong H^1(G, \mathbb{F}_p).$$

It follows that $d(G) = \dim_{\mathbb{F}_p} H^1(G, \Lambda_G)_G = f(G)$. This proves (i).

If G is not free, then $t \geq 1$. From theorem 2 it follows that $h^1(H_i) = 0$, $i = 1, \dots, t$. Since Λ_G is Λ_{H_i} -projective, the functor $\Lambda_G \hat{\otimes}_{\Lambda_{H_i}} -$ is exact, and we obtain the exact sequence

$$(\dagger) \quad 0 \longrightarrow J_{H_i} \longrightarrow \Lambda_G \longrightarrow \Lambda_G \hat{\otimes}_{\Lambda_{H_i}} \mathbb{F}_p \longrightarrow 0.$$

Using (1.5)(ii) and the assumption that G is torsion-free and so H_i is not finite, we have

$$\text{Hom}_{\Lambda_G}(\Lambda_G \hat{\otimes}_{\Lambda_{H_i}} \mathbb{F}_p, \Lambda_G) \cong \text{Hom}_{\Lambda_{H_i}}(\mathbb{F}_p, \text{Res}_{H_i} \Lambda_G) = 0$$

and

$$\text{Ext}_{\Lambda_G}^1(\Lambda_G \hat{\otimes}_{\Lambda_{H_i}} \mathbb{F}_p, \Lambda_G) \cong \text{Ext}_{\Lambda_{H_i}}^1(\mathbb{F}_p, \text{Res}_{H_i} \Lambda_G) = H^1(H_i, \text{Res}_{H_i} \Lambda_G) = 0.$$

Therefore the exact sequence (†) yields the isomorphism

$$\Lambda_G \xrightarrow[\sim]{\tilde{\phi}_{H_i}} \operatorname{Hom}_{\Lambda_G}(J_{H_i}, \Lambda_G),$$

where $\tilde{\phi}_{H_i} = \Lambda_G \hat{\otimes}_{\Lambda_{H_i}} \phi_{H_i}$. Thus J_{H_i} is indecomposable and $\operatorname{Hom}_{\Lambda_G}(I_G, \Lambda_G) \cong (\Lambda_G)^{t+r}$. Furthermore, it follows that the composite map

$$\Lambda_G \xrightarrow{\phi_G} \operatorname{Hom}_{\Lambda_G}(I_G, \Lambda_G) \xrightarrow{pr_1} \operatorname{Hom}_{\Lambda_G}(J_{H_1}, \Lambda_G)$$

is an isomorphism. Therefore, we get an isomorphism

$$(\Lambda_G)^{t-1+r} \cong \bigoplus_{i=2}^t \operatorname{Hom}_{\Lambda_G}(J_{H_i}, \Lambda_G) \oplus \operatorname{Hom}_{\Lambda_G}((\Lambda_G)^r, \Lambda_G) \xrightarrow{\sim} H^1(G, \Lambda_G).$$

The last assertion follows from the Krull-Schmidt-Azumaya theorem (1.1)(ii). \square

3 Appendix: Abstract groups

A finitely generated (abstract) group admits a decomposition into a free product of freely indecomposable groups, called its *Grushko decomposition*:

Theorem (Grushko, Kurosh): Every finitely generated group G is the free product of finitely many freely indecomposable subgroups G_i , $i = 1, \dots, s$, i.e.

$$G = \bigstar_{i=1}^s G_i.$$

This decomposition is unique in the following sense: if

$$G = \bigstar_{i=1}^n G_i = \bigstar_{j=1}^m H_j$$

are two decompositions of G into freely indecomposable subgroups, then $n = m$ and if G_i is not isomorphic to \mathbb{Z} , then there exist an element σ_i in G such that (after possibly re-ordering)

$$G_i = (H_i)^{\sigma_i}.$$

In particular, the number $s(G) = n$ of freely indecomposable factors of G is an invariant of G .

The uniqueness statement above follows from Kurosh' subgroup theorem.

Theorem (Kurosh): Let

$$G = \bigstar_{i=1}^s G_i.$$

be a decomposition of the finitely generated group G and let H be a subgroup of G of finite index. Then H admits a free product decomposition

$$H = \bigstar_{i=1}^s \bigstar_{s_i \in S_i} (G_i^{s_i} \cap H) * F_r,$$

where S_i are systems of representatives s_i of the double coset decomposition $G = \bigcup_{s_i \in S_i} H s_i G_i$ and F_r is a free group of the finite rank

$$r = \sum_{i=1}^s [(G : H) - \#S_i] - (G : H) + 1.$$

Let R be an arbitrary principal ideal domain.

Theorem: Let G be a finitely generated torsion-free group.

(i) *(Hopf): The R -module $H^1(G, R[G])$ is a free of rank 0, 1 or ∞ and $\text{rank}_R H^1(G, R[G]) = 1$ if and only if $G = \mathbb{Z}$.*

(ii) *(Stallings): $\text{rank}_R H^1(G, R[G]) = \infty$ if and only if G is freely decomposable.*

Since $H^1(G, R[G]) \cong H^1(H, R[H])$ if H is a subgroup of G of finite index, we get the following corollary.

Corollary: Let G be a finitely generated torsion-free group and let H be a subgroup of finite index. Then H is freely decomposable if and only if G is freely decomposable.

The following theorem is an immediate consequence of the theorems above and might be well-known but we can not find it in the literature.

Theorem 3.1 *Let G be a finitely generated torsion-free group and let H be a subgroup of finite index. Then*

$$s(H) = (G : H)(s(G) - 1) + 1,$$

where $s(G)$ and $s(H)$ are the number of freely indecomposable factors of G and H , respectively.

Proof: Let $G = \bigstar_{i=1}^{s(G)} G_i$ be a decomposition of G into freely indecomposable subgroups G_i . By Kurosh' subgroup theorem we have the decomposition

$$H = \bigstar_{i=1}^{s(G)} \bigstar_{s_i \in S_i} (G_i^{s_i} \cap H) * F_r.$$

Since $G_i^{s_i} \cap H$ is a subgroup of $G_i^{s_i}$ of finite index, it is freely indecomposable, hence

$$s(H) = \sum_{i=1}^{s(G)} \#S_i + \sum_{i=1}^{s(G)} [(G : H) - \#S_i] - (G : H) + 1 = (G : H)(s(G) - 1) + 1.$$

□

Concerning the structure of $H^1(G, R[G])$ one knows that this R -module is isomorphic to 0 or R or $\bigoplus_1^\infty R$, if G is an infinite finitely generated group. Now we consider this cohomology group with its right $R[G]$ -module structure given by right multiplication on $R[G]$. Again we define

$$f(G) = \text{rank}_R H^1(G, R[G])_G$$

(we will see that $f(G)$ does not depend on R).

Theorem 3.2 *Let G be a finitely generated torsion-free group. Let*

$$G = \bigast_{i=1}^{s(G)} G_i.$$

be a decomposition of G as free product, where G_i , $i = 1, \dots, s(G)$, are freely indecomposable subgroups. Then we have the following assertions.

(i) *If G is free, then $s(G) = f(G)$, and there is an exact sequence*

$$0 \longrightarrow R[G] \longrightarrow R[G]^{f(G)} \longrightarrow H^1(G, R[G]) \longrightarrow 0.$$

(ii) *If G is not free, then $s(G) = f(G) + 1$ and the right $R[G]$ -module $H^1(G, R[G])$ is free of rank $f(G)$.*

Proof: Since G is infinite, the exact sequence

$$0 \longrightarrow I_G \longrightarrow R[G] \longrightarrow R \longrightarrow 0$$

yields the exact sequence

$$0 \longrightarrow R[G] \xrightarrow{\phi} \text{Hom}_{R[G]}(I_G, R[G]) \longrightarrow H^1(G, R[G]) \longrightarrow 0,$$

where $\phi(\xi) : x \mapsto \xi x$. Using [5] (4.7), we have

$$I_G = \bigoplus_{i=1}^{s(G)} J_{G_i},$$

where $J_{G_i} = R[G] \otimes_{R[G_i]} I_{G_i}$.

If G is free of rank $s(G)$, then $I_G \cong R[G]^{s(G)}$, as $I_{\mathbb{Z}} \cong R[\mathbb{Z}]$, and we obtain the exact sequence

$$0 \longrightarrow R[G] \longrightarrow R[G]^{s(G)} \longrightarrow H^1(G, R[G]) \longrightarrow 0.$$

Since G is a duality group with dualizing module $H^1(G, \mathbb{Z}[G])$, see [2] (5.1), we obtain

$$H_0(G, H^1(G, R[G])) \cong H^1(G, R) \cong R^{s(G)}.$$

It follows that $s(G) = f(G)$.

If G is not free, then at least one factor G_i is not isomorphic to \mathbb{Z} , say G_1 . From Stallings' theorem it follows that $H^1(G_1, R[G_1]) = 0$. Since $R[G]$ is $R[G_1]$ -projective, the functor $R[G] \otimes_{R[G_1]} -$ is exact, and we obtain the exact sequence

$$(\dagger) \quad 0 \longrightarrow J_{G_1} \longrightarrow R[G] \longrightarrow R[G] \otimes_{R[G_1]} R \longrightarrow 0.$$

Using Frobenius reciprocity, we have

$$\mathrm{Hom}_{R[G]}(R[G] \otimes_{R[G_1]} R, R[G]) \cong \mathrm{Hom}_{R[G_1]}(R, \mathrm{Res}_{G_1} R[G]) = 0$$

and

$$\mathrm{Ext}_{R[G]}^1(R[G] \otimes_{R[G_1]} R, R[G]) \cong \mathrm{Ext}_{R[G_1]}^1(R, \mathrm{Res}_{G_1} R[G]) = H^1(G_1, \mathrm{Res}_{G_1} R[G]) = 0.$$

From the exact sequence (\dagger) we obtain the isomorphism

$$R[G] \xrightarrow[\sim]{\phi} \mathrm{Hom}_{R[G]}(J_{G_1}, R[G]),$$

and so a commutative and exact diagram

$$\begin{array}{ccccccc} & & \mathrm{Hom}_{R[G]}(\bigoplus_{i=2}^{s(G)} J_{G_i}, R[G]) & \longrightarrow & H^1(G, R[G]) & & \\ & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & R[G] & \longrightarrow & \mathrm{Hom}_{R[G]}(I_G, R[G]) & \longrightarrow & H^1(G, R[G]) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \\ & & R[G] & \xrightarrow{\sim} & \mathrm{Hom}_{R[G]}(J_{G_1}, R[G]) & & \end{array}$$

Since $\mathrm{Hom}_{R[G]}(J_{G_i}, R[G]) \cong R[G]$ for all factors G_i (obviously for free factors and for non-free be the consideration above), we get the isomorphism

$$H^1(G, R[G]) \cong \bigoplus_{i=2}^{s(G)} \mathrm{Hom}_{R[G]}(J_{G_i}, R[G]) \cong R[G]^{s(G)-1}.$$

This finishes the proof of the theorem. \square

Since in the situation of abstract infinite groups the Krull-Schmidt-Azumaya theorem does not hold in general, and furthermore the $R[G]$ -module $R[G]$ is not necessarily indecomposable, we have not the full analog to the assertion (2.5).

References

- [1] Anderson, F.W., Fuller, K.R. *Rings and Categories of Modules*. 2nd edition, Springer 1992
- [2] Bieri, R., Eckmann, B. *Groups with Homological Duality Generalizing Poincaré Duality*. Invent. math. **20** (1973) 103-124
- [3] Brumer, A. *Pseudocompact algebras, profinite groups and class formations*. J. of Algebra **4** (1966) 442–470
- [4] Cartan, E., Eilenberg, S. *Homological Algebra*. Princeton Math. Ser. **19**, Princeton 1956
- [5] Cohen, D.E. *Groups of Comological Dimension One*. Lect. Notes in Math. **245**, Springer Berlin-Heidelberg-New York 1972
- [6] Curtis, C. W., Reiner, I. *Methods of Representation Theory Vol I*. Wiley-Interscience Publication New York, Chichester, Brisbane, Toronto 1981
- [7] Haran, D. *On closed subgroups of free products of profinite groups*. Proc. Lond. Math. Soc. **55** (1987), 266-298
- [8] Herfort, W.N., Ribes, L. *Subgroups of free pro- p products*. Math. Proc. Camb. Phil. Soc. **101** (1987), 197-206
- [9] Korenev, A. A. *Pro- p Groups with Finite Number of Ends*. Math. Notes **76** (2004) 490–496
- [10] MacQuarrie, J. *Modular representations of profinite groups* J. pure and applied algebra **215** (2011) 753–763
- [11] Melnikov, O.V. *Subgroups and homologies of free products of profinite groups*. Math. USSR Izv. **34** (1990) 97-119
- [12] Neukirch, J., Schmidt, A., Wingberg, K. *Cohomology of Number Fields*. 2nd edition, Springer 2008
- [13] Pletch, A. *Profinite duality groups II*. J. Pure Applied Algebra **16** 1980, 285–297
- [14] Ribes, L., Zalesskiĭ, P. A. *Profinite Groups*. 2nd edition, Springer 2010
- [15] Serre, J.-P. *Sur la dimension cohomologique des groupes profinis*. Topology **3** (1965) 413–420
- [16] Stallings, J. *On torsion-free groups with infinitely many ends*. Ann. of Math. **88** (1968) 312–334

- [17] Swan, R., G. *Groups of Cohomological Dimension One.* J. of Algebra **12** (1969) 585–601
- [18] Symonds, P. *On the construction of permutation complexes for profinite groups.* Geometry & Topology Monographs **11** (2007) 369–378
- [19] Weigel, Th., Zalesskii, P. A. *Stallings' decomposition theorem for finitely generated pro- p groups.* Preprint 2013

Mathematisches Institut
 der Universität Heidelberg
 Im Neuenheimer Feld 288
 69120 Heidelberg
 Germany

e-mail: wingberg at mathi dot uni-heidelberg dot de